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The change-of-variance function of M-estimators of scale under general contamination

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Abstract

In this paper we derive the change-of-variance function of M-estimators of scale under general contamination, thereby extending the formula in Hampel et al. (1986). We say that an M-estimator is B-robust if its influence function is bounded, and we call it V-robust if its change-of-variance function is bounded from above. It is shown, for a natural class of M-estimators, that the general notion of V-robustness still implies B-robustness. Several classes of M-estimators are studied closely, as well as some typical examples and their interpretation.

Keywords: Influence function; Change-of-variance function; B-robustness; V-robustness

1. Introduction

The *influence function* $\mathrm{IF}(x,S,F)$ of a statistical functional S at a distribution F is defined as the kernel of a first-order von Mises derivative:

$$\int \mathrm{IF}(x,S,F)\,\mathrm{d}G(x) = \frac{\partial}{\partial\varepsilon}[S((1-\varepsilon)F + \varepsilon G)]_{\varepsilon=0}, \tag{1.1}$$

where G ranges over all distributions (including point masses). Analogously, the *change-of-variance function* [3] is defined by

$$\int \mathrm{CVF}(x,S,F)\,\mathrm{d}G(x) = \frac{\partial}{\partial\varepsilon}[V(S,(1-\varepsilon)F + \varepsilon G)]_{\varepsilon=0}, \tag{1.2}$$

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where $V(S,F)$ is the asymptotic variance of S at F. The latter formula [1, p. 128] was applied to M-estimators of location, but for M-estimators of scale only distributions G with $S(G) = S(F) = 1$ were used for simplicity. In the present paper we derive the change-of-variance function for M-estimators of scale under general contaminating distributions G.

Let us recall the definition of an M-estimator of scale. Suppose we have one-dimensional observations $X_1, \ldots, X_n$ which are independent and identically distributed according to a distribution from the *parametric model* $\{F_\sigma; \sigma > 0\}$, where $F_\sigma(x) = F(x/\sigma)$. An M-estimator $S_n(X_1, \ldots, X_n)$ of σ is given by

$$\sum_{i=1}^{n} \chi(X_i/S_n) = 0$$

and corresponds to the statistical functional S defined by

$$\int \chi(x/S(F))\,\mathrm{d}F(x) = 0. \tag{1.3}$$

The influence function of S is

$$\mathrm{IF}(u,S,F) = \frac{\chi(u/S(F))\,S^2(F)}{\int x\chi'(x/S(F))\,\mathrm{d}F(x)}. \tag{1.4}$$

For more information, see [1]. An important summary value of the influence function is the *gross-error sensitivity* of S at F, defined by

$$\gamma^* = \sup_u |\mathrm{IF}(u,S,F)|. \tag{1.5}$$

It measures the worst influence that a small amount of contamination can have on the value of the estimator. Therefore, a desirable feature is that γ^* be finite, in which case S is called *B-robust* (bias-robust) at F.

Under certain regularity conditions, $\sqrt{n}(S_n - \sigma)$ is asymptotically normal with asymptotic variance

$$V(S,F) = \int \mathrm{IF}^2(u,S,F)\,\mathrm{d}F(u)$$
$$= \frac{\int \chi^2(u/S(F))\,S^4(F)\,\mathrm{d}F(u)}{(\int x\,\chi'(x/S(F))\,\mathrm{d}F(x))^2}. \tag{1.6}$$

The change-of-variance function is then found by inserting (1.6) in (1.2), and the resulting expression will be given in Section 2. We then define the *change-of-variance sensitivity* κ^* as $+\infty$ if a delta function with positive factor occurs in the CVF, and otherwise as

$$\kappa^* = \sup_z \frac{\mathrm{CVF}(z,S,F)}{V(S,F)}. \tag{1.7}$$

Note that large negative values of the CVF merely point to a decrease in V, indicating a better accuracy. If κ^* is finite then S is called *V-robust* (variance-robust) at F.

2. The change-of-variance function of M-estimators of scale

Recall that $F_\sigma(x) = F(x/\sigma)$. We need the following regularity conditions on F:

(F1) F has a twice continuously differentiable density f (with respect to the Lebesgue measure λ) which is symmetric around zero and satisfies $f(x) > 0\ \forall x \in \mathbb{R}$.

(F2) The mapping $\Lambda = -f'/f = (-\ln f)'$ satisfies $\Lambda'(x) > 0\ \ \forall x \in \mathbb{R}$, and $\int \Lambda' f \,\mathrm{d}\lambda = -\int \Lambda f' \,\mathrm{d}\lambda < \infty$.

Let us denote

$$A(\chi) = \int \chi^2(x)\,\mathrm{d}F(x), \tag{2.1}$$

$$B(\chi) = \int x\chi'(x)\,\mathrm{d}F(x). \tag{2.2}$$

We will assume that χ belongs to the class Ψ of all functions satisfying the following four regularity conditions:

(R1) χ is well-defined and continuous on $\mathbb{R}\backslash D^{(0)}(\chi)$, where $D^{(0)}(\chi)$ is finite. In each point of $D^{(0)}(\chi)$ there exist finite left and right limits of χ which are different. Also $\chi(-x) = \chi(x)$ if $\{-x, x\} \subset \mathbb{R}\backslash D^{(0)}(\chi)$, and there exists $d > 0$ such that $\chi(x) \leqslant 0$ on $(0, d)$ and $\chi(x) \geqslant 0$ on (d, ∞).

(R2) The set $D^{(1)}(\chi)$ of points in which χ is continuous but in which χ' is not defined or not continuous, is finite.

(R3) $\int \chi(x)\mathrm{d}F(x) = 0$ (Fisher consistency) and $0 < A(\chi) < \infty$.

(R4) $0 < B(\chi) = \int (x\Lambda(x) - 1)\chi(x)\mathrm{d}F(x) < \infty$.

From (1.2) and (1.6) we obtain

$$\begin{aligned}
\mathrm{CVF}(z,S,F) = {}& \left(\int \chi'(x/S(F))x\,\mathrm{d}F(x)\right)^{-3} \Bigg[\left(\int \chi'(x/S(F))x\,\mathrm{d}F(x)\right) \\
&\times\Bigg(-\int \chi^2(u/S(F))S^4(F)\mathrm{d}F(u) + \chi^2(z/S(F))S^4(F) \\
&\quad - 2\mathrm{IF}(z,S,F)\int \chi(u/S(F))\chi'(u/S(F))(u/S^2(F))S^4(F)\mathrm{d}F(u) \\
&\quad + 4\mathrm{IF}(z,S,F)\int \chi^2(u/S(F))S^3(F)\mathrm{d}F(u)\Bigg) \\
&- 2\left(\int \chi^2(u/S(F))S^4(F)\mathrm{d}F(u)\right) \\
&\times\Bigg(-\int \chi'(x/S(F))x\,\mathrm{d}F(x) + (\chi'(z/S(F))z \\
&\quad - \mathrm{IF}(z,S,F)\int \chi''(x/S(F))(x/S^2(F))x\,\mathrm{d}F(x)\Bigg)\Bigg].
\end{aligned} \tag{2.3}$$

Making use of (2.1), (2.2), and $S(F) = 1$ at the model distribution, (2.3) becomes

$$\mathrm{CVF}(z,S,F) = \frac{A(\chi)}{B^2(\chi)}\left[1 + \frac{\chi^2(z)}{A(\chi)} - 2\frac{z\chi'(z)}{B(\chi)} + C(\chi)\frac{\chi(z)}{B(\chi)}\right], \tag{2.4}$$

where

$$C(\chi) = 4 - \frac{2}{A(\chi)}\int u\chi(u)\chi'(u)\mathrm{d}F(u) + \frac{2}{B(\chi)}\int u^2\chi''(u)\mathrm{d}F(u). \tag{2.5}$$

Note that (2.4) differs from the expression in [1] by the addition of the last term, the integral of which is zero when $S(G) = 1$. This distinction does not exist for location, at least in the case of odd ψ, as can be seen in [1, pp. 145–146], where

$$\tilde{C}(\psi) = 2\int\left(\frac{\psi''(u)}{B(\psi)} - \frac{\psi(u)\psi'(u)}{A(\psi)}\right)\mathrm{d}F(u) = 0.$$

From here on we will assume that $C(\chi) \geqslant 0$, which is true in all practical applications. In Section 4.2 we will derive an alternative expression for $C(\chi)$ which is easier to compute than (2.5).

3. Relation between B-robustness and V-robustness

Let us define

$$\gamma^- = \sup_{u\in(0,d)} (-\mathrm{IF}(u,S,F)), \tag{3.1}$$

$$\gamma^+ = \sup_{u\in(d,+\infty)} \mathrm{IF}(u,S,F)). \tag{3.2}$$

In the theorems below we will impose that $\gamma^+ \geqslant \gamma^-$ (and hence $\gamma^* = \gamma^+$). This is a very natural requirement for scale estimators. For instance, when discussing breakdown properties [2], notes that $\gamma^+ \geqslant \gamma^-$ in the more interesting cases. The opposite situation leads to implosion of the scale estimator, as well as to lower efficiency.

The first theorem shows that the concept of V-robustness is stronger than the concept of B-robustness.

Theorem 1. *For all $\chi \in \Psi$ with $\gamma^+ \geqslant \gamma^-$ and $C(\chi) \geqslant 0$, V-robustness implies B-robustness. In fact*

$$\gamma^* \leqslant \tfrac{1}{2}\left[\sqrt{V^2(S,F)C^2(\chi) + 4V(S,F)(\kappa^* - 1)} - V(S,F)C(\chi)\right].$$

Proof. Suppose that κ^* is finite and that there exists some x_0 for which

$$|\mathrm{IF}(x_0,S,F)| > \tfrac{1}{2}\left[\sqrt{V^2(S,F)C^2(\chi) + 4V(S,F)(\kappa^* - 1)} - V(S,F)C(\chi)\right].$$

Without loss of generality, put $x_0 \notin D^{(1)}(\chi)$ and $x_0 > d$. It follows that

$$\chi(x_0) > \frac{1}{2}\left[\sqrt{\left(\frac{A(\chi)C(\chi)}{B(\chi)}\right)^2 + 4A(\chi)(\kappa^* - 1)} - \frac{A(\chi)C(\chi)}{B(\chi)}\right] = b.$$

If $\chi'(x_0) \leqslant 0$ then

$$1 + \frac{\chi^2(x_0)}{A(\chi)} - 2\frac{x_0\chi'(x_0)}{B(\chi)} + \frac{C(\chi)}{B(\chi)}\chi(x_0) \geqslant 1 + \frac{b^2}{A(\chi)} + \frac{C(\chi)}{B(\chi)}b = \kappa^*,$$

a contradiction. Therefore, $\chi'(x_0) > 0$. Since we have $\chi(x_0) > 0$, there exists $\varepsilon > 0$ such that $\chi'(t) > 0$ for all t in $[x_0, x_0 + \varepsilon)$, so $\chi(x) > \chi(x_0)$ for all x in $(x_0, x_0 + \varepsilon]$. It follows that $\chi(x) > \chi(x_0) > b$ for all $x > x_0, x \notin D^{(0)}(\chi)$ because only upward jumps of χ are allowed for positive x. As $D^{(0)}(\chi) \cup D^{(1)}(\chi)$ is finite, we may assume that $[x_0, +\infty) \cap (D^{(0)}(\chi) \cup D^{(1)}(\chi))$ is empty. It holds that

$$1 + \frac{\chi^2(x)}{A(\chi)} - 2\frac{x\chi'(x)}{B(\chi)} + \frac{C(\chi)}{B(\chi)}\chi(x) \leqslant \kappa^*.$$

Therefore

$$\chi^2(x) - 2x\chi'(x)\frac{A(\chi)}{B(\chi)} \leqslant A(\chi)(\kappa^* - 1) - \frac{C(\chi)A(\chi)}{B(\chi)}\chi(x) \leqslant A(\chi)(\kappa^* - 1) - \frac{C(\chi)A(\chi)}{B(\chi)}b \leqslant b^2,$$

hence

$$\chi^2(x) - 2x\chi'(x)\frac{A(\chi)}{B(\chi)} \leqslant b^2$$

for all $x \geqslant x_0$. Hence

$$\frac{\chi'(x)}{\chi^2(x) - b^2} \geqslant \frac{B(\chi)}{2A(\chi)}\frac{1}{x}.$$

Putting

$$R(x) = -\frac{1}{b}\coth^{-1}\left(\frac{\chi(x)}{b}\right)$$

and

$$P(x) = \frac{B(\chi)}{2A(\chi)}\ln(x),$$

it follows that $R'(x) \geqslant P'(x)$ for all $x \geqslant x_0$. Hence $R(x) - R(x_0) \geqslant P(x) - P(x_0)$, and thus

$$\coth^{-1}\left(\frac{\chi(x)}{b}\right) \leqslant b\left[P(x_0) - R(x_0) - \frac{B(\chi)}{2A(\chi)}\ln(x)\right].$$

However, the left member is positive because $\chi(x) > b$ and the right member tends to $-\infty$ for $x \to \infty$, a contradiction. This proves the desired inequality. □

Theorem 2. *For all $\chi \in \Psi$ with $\gamma^+ \geqslant \gamma^-$ and $C(\chi) \geqslant 0$, and χ nondecreasing for $x \geqslant 0$, V-robustness and B-robustness are equivalent. In fact*

$$\kappa^* = 1 + \frac{(\gamma^*)^2}{V(S,F)} + C(\chi)\gamma^*.$$

Proof. One of the two inequalities follows from Theorem 1. For the other, assume that S is B-robust. Because χ is monotone, the CVF can only contain negative delta functions, which do not contribute to κ^*. For all $x \geqslant 0$ it holds that $\chi'(x) \geqslant 0$, so

$$1 + \frac{\chi^2(x)}{A(\chi)} - 2\frac{x\chi'(x)}{B(\chi)} + \frac{C(\chi)}{B(\chi)}\chi(x) \leqslant 1 + \frac{(\gamma^*)^2}{V(S,F)} + C(\chi)\gamma^*.$$

Hence, S is also V-robust. □

Theorem 3. *For all $\chi \in \Psi$ with $\gamma^+ \geqslant \gamma^-$ and $C(\chi) \geqslant 0$, and χ nondecreasing for $x \geqslant 0$, we have*

$$\kappa^* \geqslant 2 + C(\chi)\gamma^*.$$

Proof. We have

$$V(S,F) = \int \mathrm{IF}^2(u,S,F)\,\mathrm{d}F(u) \leqslant (\gamma^*)^2.$$

Using Theorem 2, it follows that

$$\kappa^* = 1 + \frac{(\gamma^*)^2}{V(S,F)} + C(\chi)\gamma^* \geqslant 2 + C(\chi)\gamma^*. \qquad \square$$

4. Examples

4.1. The L^q scale estimator

The L^q scale estimator at F is given by

$$\chi(x) = |x|^q - \int |x|^q \mathrm{d}F(x), \quad \text{with } q > 0. \tag{4.1}$$

Theorem 4. *For any distribution F and any $q > 0$, the L^q scale estimator satisfies*

$$C(\chi) = 2.$$

Proof. From $\chi'(x) = q|x|^{q-1}\,\mathrm{sign}(x)$ we deduce the two relations

$$x\chi'(x) = q\chi(x) + B(\chi)$$

and

$$x^2\chi''(x) = (q-1)x\chi'(x).$$

This yields

$$\int x\chi(x)\chi'(x)\mathrm{d}F(x) = q\int \chi^2(x)\mathrm{d}F(x) + B(\chi)\int \chi(x)\mathrm{d}F(x) = qA(\chi),$$

$$\int x^2\chi''(x)\mathrm{d}F(x) = (q-1)B(\chi).$$

Hence

$$C(\chi) = 4 - \frac{2}{A(\chi)}qA(\chi) + \frac{2}{B(\chi)}(q-1)B(\chi) = 2. \qquad \square$$

Theorem 5. *The L^q scale estimator is neither B-robust nor V-robust at any distribution F, that is to say*

$$\gamma^* = \infty \quad \textit{and} \quad \kappa^* = \infty.$$

Proof. As χ is unbounded, the estimator is not B-robust. Moreover, as the CVF behaves like x^{2q} with a positive factor when $x \to \infty$, it is not bounded from above. $\square$

The maximum likelihood estimator (MLE) at $F = \Phi$ is given by $\chi(x) = x^2 - 1$, obtained by putting $q = 2$ in (4.1). This yields

$$A(\chi) = \int \chi^2(x)\mathrm{d}\Phi(x) = 2,$$

$$B(\chi) = \int x\chi'(x)\mathrm{d}\Phi(x) = 2,$$

$$\int \chi(x)\chi'(x)x\,\mathrm{d}\Phi(x) = 4,$$

$$\int \chi''(x)x^2\,\mathrm{d}\Phi(x) = 2.$$

Hence

$$\mathrm{IF}(u,S,\Phi) = \tfrac{1}{2}(u^2-1) \quad \text{with} \quad \gamma^* = \infty,$$

$$\mathrm{CVF}(z,S,\Phi) = \tfrac{1}{4}(z^4 - 4z^2 + 1) \quad \text{with} \quad \kappa^* = \infty.$$

Both functions are plotted in Fig. 1. We see that the maximum likelihood estimator at Φ is neither B-robust nor V-robust. For $q = 1$ we obtain the mean deviation with $\chi(x) = |x| - \sqrt{2/\pi}$ which is again neither B-robust nor V-robust.

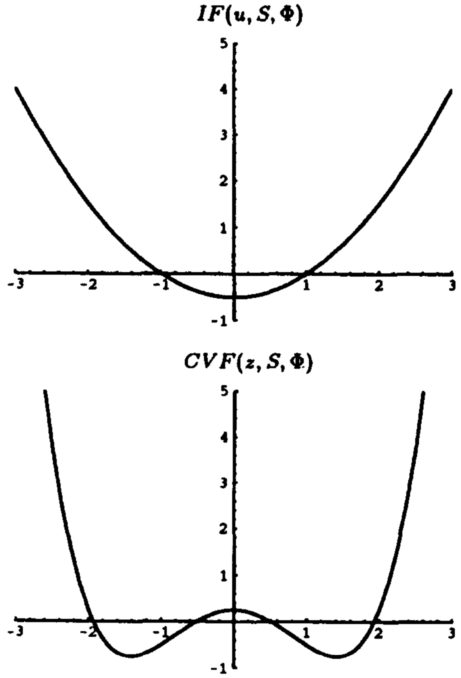

Fig. 1. The influence function and change-of-variance function of the MLE.

4.2. Computation of $C(\chi)$ at the Gaussian model

Let us recall that

$$C(\chi) = 4 - \frac{2}{A(\chi)} \int x\chi(x)\chi'(x)\mathrm{d}F(x) + \frac{2}{B(\chi)} \int x^2\chi''(x)\mathrm{d}F(x).$$

Theorem 6. *At the Gaussian distribution $F = \Phi$ we have*

$$C(\chi) = 1 - \frac{1}{A(\chi)} \int x^2\chi^2(x)\mathrm{d}\Phi(x) + \frac{2}{B(\chi)} \int (x^4 - 3x^2)\chi(x)\mathrm{d}\Phi(x).$$

Proof. Denoting the density of Φ by ϕ we find

$$\begin{aligned}\int x\chi(x)\chi'(x)\mathrm{d}\Phi(x) &= \frac{1}{2}\int x(\chi^2(x))'\phi(x)\mathrm{d}x \\ &= -\frac{1}{2}\int \chi^2(x)(\phi(x) + x\phi'(x))\mathrm{d}x \\ &= -\frac{1}{2}\int \chi^2(x)(1 - x^2)\phi(x)\mathrm{d}x \\ &= \frac{1}{2}\left(\int x^2\chi^2(x)\mathrm{d}\Phi(x) - A(\chi)\right) \qquad (4.2)\end{aligned}$$

and

$$\begin{aligned}\int x^2\chi''(x)\mathrm{d}\Phi(x) &= \int x^2(\chi'(x))'\phi(x)\mathrm{d}x\\ &= -\int(2x\phi(x)+x^2\phi'(x))\chi'(x)\mathrm{d}x\\ &= -2B(\chi)+\int x^3\chi'(x)\phi(x)\mathrm{d}x\\ &= -2B(\chi)-\int(x^3\phi(x))'\chi(x)\mathrm{d}x\\ &= -2B(\chi)+\int(x^4-3x^2)\chi(x)\mathrm{d}\Phi(x).\end{aligned} \tag{4.3}$$

This yields

$$\begin{aligned}C(\chi) &= 4-\frac{1}{A(\chi)}\left(\int x^2\chi^2(x)\mathrm{d}\Phi(x)-A(\chi)\right)+\frac{2}{B(\chi)}\left(-2B(\chi)+\int(x^4-3x^2)\chi(x)\mathrm{d}\Phi(x)\right)\\ &= 1-\frac{1}{A(\chi)}\int x^2\chi^2(x)\mathrm{d}\Phi(x)+\frac{2}{B(\chi)}\int(x^4-3x^2)\chi(x)\mathrm{d}\Phi(x). \qquad \square\end{aligned}$$

4.3. The λth absolute deviation estimator (λ-MAD) at the Gaussian model

Consider the λth absolute deviation estimator (λ-MAD) at $F=\Phi$ given by

$$\chi(x)=\begin{cases}(\lambda-1)/\lambda & \text{if } -\Phi^{-1}(\tfrac{1}{2}+\tfrac{1}{2}\lambda)<x<\Phi^{-1}(\tfrac{1}{2}+\tfrac{1}{2}\lambda),\\ 1 & \text{elsewhere,}\end{cases}$$

with $0<\lambda<1$. Let us now look at Fig. 2, where $C(\chi)$, $\gamma^*(\chi)$ and $\kappa^*(\chi)$ are plotted as functions of λ.

First of all, we see that $C(\chi)>0$ for all λ. Secondly, the gross-error sensitivity is minimal for $\lambda=\frac{1}{2}$, which corresponds to the usual median absolute deviation (MAD). Finally, the change-of-variance sensitivity tends to the value 2 as λ tends to zero. However, note that for $\lambda<\frac{1}{2}$ we do not have the condition $\gamma^+\geqslant\gamma^-$ required by the theorems of Section 3.

Consider the special case of $\lambda=\frac{1}{2}$, which corresponds to the usual median absolute deviation at $F=\Phi$, given by $\chi(x)=\operatorname{sign}(|x|-q)$ where $q=\Phi^{-1}(3/4)$. This yields

$$A(\chi)=\int\chi^2(x)\mathrm{d}\Phi(x)=1,$$

$$B(\chi)=\int x\chi'(x)\mathrm{d}\Phi(x)=4q\phi(q),$$

$$\int x^2\chi^2(x)\,\mathrm{d}\Phi(x)=1,$$

$$\int(x^4-3x^2)\chi(x)\,\mathrm{d}\Phi(x)=4q^3\phi(q).$$

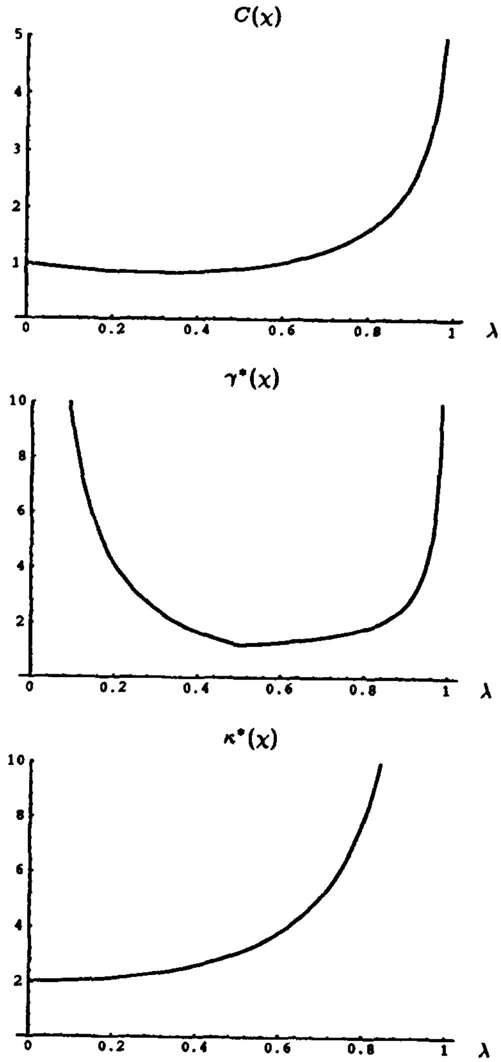

Fig. 2. The values of $C(\chi)$, $\gamma^*(\chi)$ and $\kappa^*(\chi)$ as a function of λ for the λ-MAD.

Therefore

$$\mathrm{IF}(u,S,\Phi) = \frac{\mathrm{sign}(|u|-q)}{4q\phi\,(q)} \quad \text{with } \gamma^* = \frac{1}{4q\phi(q)} = 1.166,$$

$$\mathrm{CVF}(z,S,\Phi) = \frac{1}{(4q\phi(q))^2}\left[2 - \frac{1}{q\phi(q)}(\delta_q(z) + \delta_{-q}(z)) + 2q^2\,\frac{\mathrm{sign}(|z|-q)}{4q\phi(q)}\right]$$

$$\text{with } \kappa^* = 2 + \frac{q}{2\phi(q)} = 3.061.$$

The MAD at Φ is thus both B-robust and V-robust (see Fig. 3).

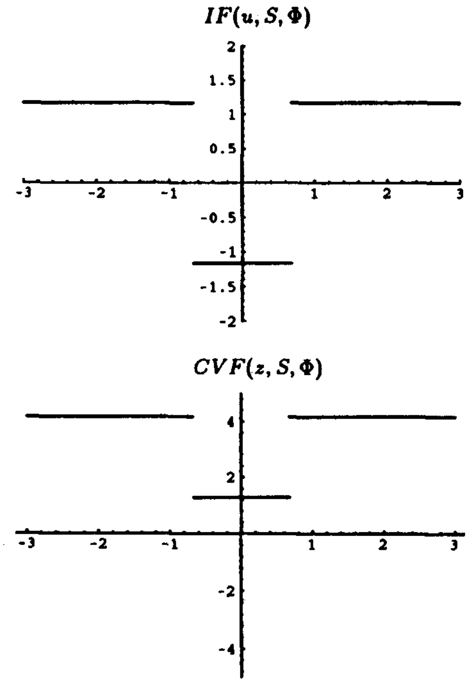

Fig. 3. The influence function and change-of-variance function of the MAD.

4.4. *The Welsch estimator at the Gaussian model*

Let us consider the Welsch estimator family at $F = \Phi$ given by

$$\chi(x) = \int \exp\left(-\frac{x^2}{d}\right) d\Phi(x) - \exp\left(-\frac{x^2}{d}\right) \quad \text{with } d > 0,$$

and let us look at the graphs of $C(\chi)$, $\gamma^*(\chi)$ and $\kappa^*(\chi)$ as functions of $d > 0$ in Fig. 4.

Also here we have $C(\chi) > 0$ for all $d > 0$. Secondly, the gross-error sensitivity is minimal for $d = 0.666$ which corresponds to the case $\gamma^* = \gamma^+ = \gamma^-$. Finally, the change-of-variance sensitivity is smallest for $d = 0.190$, which corresponds to a case where $\gamma^+ < \gamma^- = \gamma^*$.

5. Conclusions

In this paper we have derived the change-of-variance function of M-estimators of scale under general contamination, in which case the additional term $V(\chi)\,C(\chi)\,\mathrm{IF}(z)$ arises. We have seen that it is still true that V-robustness implies B-robustness. The L^q scale estimators, which have a constant $C(\chi)$, are neither B-robust nor V-robust. An alternative formula for $C(\chi)$ has been obtained, and used to analyze the λ-MAD and the Welsch estimators.

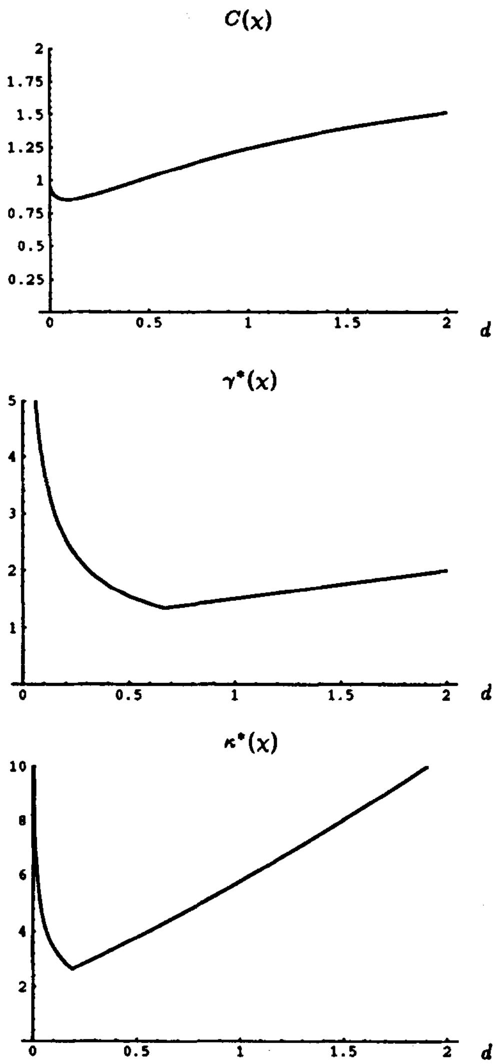

Fig. 4. The values of $C(\chi)$, $\gamma^*(\chi)$ and $\kappa^*(\chi)$ as a function of d for the Welsch estimator.

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