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The correlation structure of the sample autocovariance function for a particular class of time series with elliptically contoured distribution

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Abstract

In the context of time series, the classical estimator of the autocovariance function can be written as a quadratic form of the observations. If data have an elliptically contoured distribution with constant mean, then the correlation between the sample autocovariance function at two different lags is a function of the time design matrix and the covariance matrix of the process. When data have a regular support, an explicit formula for this correlation is available for a particular family of covariance matrices. Surprisingly, this correlation structure is exactly the same as the one for a Gaussian white noise.

Keywords: Time series; Dependent data; Autocovariance; Multivariate distribution; Kurtosis

1. Introduction

The autocovariance function plays an important role in time series analysis. For example, it is often used to study the underlying dependence structure of the process (Box and Jenkins, 1976; Brockwell and Davis, 1987). This is an important step towards constructing an appropriate mathematical model for the data. However, estimates of the sample autocovariance function at different time lags are correlated, for the same observation is used for different lags. This is true, even if data are independent. The case of Gaussian white noise has been studied by Dufour and Roy (1985, 1989), Anderson (1990, 1991, 1993), and Anderson and Chen (1996). Some extensions to spherical distributions can be found in Dufour and Roy (1985, 1989). In this paper, we study the correlation structure of the sample autocovariance function and compute an explicit formula for a particular family of covariance matrices of the process with elliptically contoured distribution, thus extending the results in Dufour and Roy (1985, 1989), and Anderson (1990). Moreover, our approach is based on generalized multivariate statistical analysis, and yields results which do not depend on the underlying dependence structure of the process, but only on the sample size and the lag distance. Further studies, for

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example in the case of ARIMA processes, may be found in De Gooijer (1980), Anderson (1982), as well as Anderson and De Gooijer (1983, 1988).

Consider a time series $\{X_t : t \in \mathbb{Z}\}$ and assume that it satisfies the hypothesis of second-order stationarity:
(i) $\mathrm{E}(X_t^2) < \infty$, $\forall t \in \mathbb{Z}$,
(ii) $\mathrm{E}(X_t) = \mu = \text{constant}$, $\forall t \in \mathbb{Z}$,
(iii) $\mathrm{Cov}(X_{t+h}, X_t) = \gamma(h)$, $\forall t, h \in \mathbb{Z}$,
where $\gamma(h)$ is the autocovariance function of X_t at lag h. The classical estimator for the autocovariance function, based on the method-of-moments, is

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{i=1}^{n-h} (X_{i+h} - \bar{X})(X_i - \bar{X}), \quad 0 \leqslant h \leqslant n-1, \tag{1}$$

where $\bar{X} = 1/n \sum_{i=1}^{n} X_i$. The simple form of this estimator allows us to write (1) as a quadratic form. In fact, if $\boldsymbol{x} = (X_1, \ldots, X_n)^{\mathrm{T}}$ is the data vector and $D(h)$ is the time design matrix of the data at lag h, then

$$\hat{\gamma}(h) = (1/n)\, \boldsymbol{x}^{\mathrm{T}} M D(h) M \boldsymbol{x}, \tag{2}$$

where $M = I_n - (1/n)\mathbf{1}_n \mathbf{1}_n^{\mathrm{T}}$ is a symmetric matrix satisfying $M^2 = M$, I_n is the identity matrix of size $n \times n$, and $\mathbf{1}_n = (1, \ldots, 1)^{\mathrm{T}} \in \mathbb{R}^n$. It is then straightforward to compute the first and second moments of the above expression (2), as shown in the next theorem.

Let us first recall some concepts on elliptically contoured distributions (Fang et al., 1989; Fang and Zhang, 1990; Fang and Anderson, 1990). A random vector $\boldsymbol{x} \in \mathbb{R}^n$ is said to have an elliptically contoured distribution $\mathrm{EC}_n(\boldsymbol{\mu}, \Sigma^*, \phi)$ if its characteristic function has the form

$$\mathrm{e}^{\mathrm{i}\boldsymbol{t}^{\mathrm{T}}\boldsymbol{\mu}} \phi(\boldsymbol{t}^{\mathrm{T}} \Sigma^* \boldsymbol{t}), \tag{3}$$

where $\mathrm{i} = \sqrt{-1}$, $\boldsymbol{t} \in \mathbb{R}^n$, $\boldsymbol{\mu} \in \mathbb{R}^n$, $\Sigma^* \in \mathbb{R}^{n \times n}$ is a positive-definite matrix, and ϕ a real function such that (3) be a characteristic function. The expectation of $\boldsymbol{x}$ is $\mathrm{E}(\boldsymbol{x}) = \boldsymbol{\mu}$ and its covariance matrix is $\mathrm{Var}(\boldsymbol{x}) = \Sigma = -2\phi'(0)\Sigma^*$. This is a general class of distributions whose contours of equal density have the same elliptical shape as the multivariate Gaussian, but which contains long-tailed and short-tailed distributions. Some important subclasses of elliptically contoured distributions are the Kotz-type, Pearson-type, multivariate t, multivariate Cauchy, multivariate Bessel, logistic, scale mixture, and of course the multivariate Gaussian with $\phi(u) = \mathrm{e}^{-u/2}$. If $\boldsymbol{\mu} = \mathbf{0}$ and $\Sigma^* = I_n$, then $\boldsymbol{x}$ has a spherical distribution, and the contours of equal density have a circular shape. Recently, some Q–Q probability plots were proposed (Li et al., 1997) to test spherical and elliptical symmetry in the data. Muirhead (1982) defines the kurtosis parameter κ of an elliptically contoured distribution $\mathrm{EC}_n(\boldsymbol{\mu}, \Sigma^*, \phi)$ as

$$\kappa = \frac{\phi''(0)}{(\phi'(0))^2} - 1,$$

where $\phi'(0)$ and $\phi''(0)$ are the first and the second derivatives of ϕ, evaluated at zero. In particular, the kurtosis parameter is equal to zero for the multivariate Gaussian distribution. Subsequently, we focus on elliptically contoured distributions with kurtosis parameter $\kappa = 0$, because they yield the same correlation structure of the autocovariance function (2) as for the multivariate Gaussian distribution. This is stated in the following theorem.

Theorem 1. *Let $\boldsymbol{x}$ be a random vector with an elliptically contoured distribution $\mathrm{EC}_n(\boldsymbol{\mu}, \Sigma^*, \phi)$, where $\boldsymbol{\mu} = \mu\mathbf{1}_n$ and the kurtosis parameter is $\kappa = 0$. Then, the sample autocovariance function $\hat{\gamma}(h)$ satisfies:*
(a) $\mathrm{E}(\hat{\gamma}(h)) = \dfrac{1}{n}\mathrm{tr}[MD(h)M\Sigma^*]$,

(b) $\mathrm{Var}(\hat{\gamma}(h)) = \dfrac{2}{n^2}\mathrm{tr}[MD(h)M\Sigma^* MD(h)M\Sigma^*]$,

(c) $\mathrm{Cov}(\hat{\gamma}(h_1), \hat{\gamma}(h_2)) = \dfrac{2}{n^2}\mathrm{tr}[MD(h_1)M\Sigma^* MD(h_2)M\Sigma^*]$,

(d) $\mathrm{Corr}(\hat{\gamma}(h_1), \hat{\gamma}(h_2)) = \dfrac{\mathrm{tr}[MD(h_1)M\Sigma^* MD(h_2)M\Sigma^*]}{\sqrt{\mathrm{tr}[MD(h_1)M\Sigma^* MD(h_1)M\Sigma^*]\mathrm{tr}[MD(h_2)M\Sigma^* MD(h_2)M\Sigma^*]}}$,

where tr[.] *is the trace operator.*

Proof. These results are automatic by-products of multivariate analysis of quadratic forms for elliptically contoured distributions (Li, 1987) and the property $M\mathbf{1}_n = \mathbf{0}$. □

In particular, note that the multivariate Gaussian distribution satisfies this theorem.

2. A particular family of covariance matrices

Suppose that the covariance matrix $\Sigma \in \mathbb{R}^n$ of the data belongs to the particular family $\mathscr{S}$ of matrices:

$$\mathscr{S} = \{\Sigma \mid \Sigma = \alpha I_n + \mathbf{1}_n \boldsymbol{a}^{\mathrm{T}} + \boldsymbol{a}\mathbf{1}_n^{\mathrm{T}}\}, \tag{4}$$

where $\alpha \in \mathbb{R}$ and $\boldsymbol{a} = (a_1, \ldots, a_n)^{\mathrm{T}} \in \mathbb{R}^n$ are defined in such a way that Σ is positive definite. For instance, straightforward computations show that the eigenvalues of a covariance matrix $\Sigma \in \mathscr{S}$ are α with multiplicity $n-2$ and

$$\alpha + \sum_{i=1}^n a_i \pm \left(n \sum_{i=1}^n a_i^2\right)^{1/2}.$$

Therefore, for any vector $\boldsymbol{a} = (a_1, \ldots, a_n)^{\mathrm{T}} \in \mathbb{R}^n$, one can choose $\alpha > 0$ such that

$$\alpha > \left(n \sum_{i=1}^n a_i^2\right)^{1/2} - \sum_{i=1}^n a_i \tag{5}$$

in order to insure positive definiteness of Σ. For this particular family $\mathscr{S}$ of matrices, formula (d) of Theorem 1 for the correlation reduces to the expression given in the next theorem.

Theorem 2. *Let $\boldsymbol{x}$ be a random vector with an elliptically contoured distribution* $\mathrm{EC}_n(\boldsymbol{\mu}, \Sigma^*, \phi)$, *where* $\boldsymbol{\mu} = \mu\mathbf{1}_n$, $\Sigma = -2\phi'(0)\Sigma^* \in \mathscr{S}$, *and the kurtosis parameter is* $\kappa = 0$. *Then, the correlation of the sample autocovariance function is*

$$\mathrm{Corr}(\hat{\gamma}(h_1), \hat{\gamma}(h_2)) = \frac{\mathrm{tr}[MD(h_1)MD(h_2)]}{\sqrt{\mathrm{tr}[MD(h_1)MD(h_1)]\mathrm{tr}[MD(h_2)MD(h_2)]}}, \tag{6}$$

which depends only on the time design matrix $D(h)$ and the matix M.

Proof. The result is a direct by-product of the following computation:

$$\begin{aligned}&\mathrm{tr}[MD(h_1)M\Sigma^* MD(h_2)M\Sigma^*]\\ &\quad= \frac{1}{4\phi'(0)^2}\mathrm{tr}[MD(h_1)M(\alpha I_n + \mathbf{1}_n\boldsymbol{a}^{\mathrm{T}} + \boldsymbol{a}\mathbf{1}_n^{\mathrm{T}})MD(h_2)M(\alpha I_n + \mathbf{1}_n\boldsymbol{a}^{\mathrm{T}} + \boldsymbol{a}\mathbf{1}_n^{\mathrm{T}})]\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4\phi'(0)^2}\operatorname{tr}[(\alpha MD(h_1)M + MD(h_1)M\mathbf{1}_n\boldsymbol{a}^{\mathrm{T}} + MD(h_1)M\boldsymbol{a}\mathbf{1}_n^{\mathrm{T}}) \\
&\quad \times(\alpha MD(h_2)M + MD(h_2)M\mathbf{1}_n\boldsymbol{a}^{\mathrm{T}} + MD(h_2)M\boldsymbol{a}\mathbf{1}_n^{\mathrm{T}})] \\
&= \frac{1}{4\phi'(0)^2}\operatorname{tr}[(\alpha MD(h_1)M + MD(h_1)M\boldsymbol{a}\mathbf{1}_n^{\mathrm{T}})(\alpha MD(h_2)M + MD(h_2)M\boldsymbol{a}\mathbf{1}_n^{\mathrm{T}})] \\
&= \frac{\alpha^2}{4\phi'(0)^2}\operatorname{tr}[MD(h_1)MD(h_2)],
\end{aligned}
$$

by using the fact that $M\mathbf{1}_n = \mathbf{0}$ and $M^2 = M$. □

Note that the family $\mathscr{S}$ contains several interesting structures. In particular, the independent case is obtained by letting $\alpha = \sigma^2$ and $\boldsymbol{a} = \mathbf{0}$, thus yielding $\Sigma = \sigma^2 I_n$. Note that "uncorrelated" becomes "independent" for the multivariate Gaussian distribution. The equicorrelation case is also a member of $\mathscr{S}$ obtained by letting $\alpha = 1-\rho$ and $\boldsymbol{a} = (\rho/2)\mathbf{1}_n$, thus yielding

$$
\Sigma = \begin{pmatrix} 1 & \rho & \dots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \dots & \rho & 1 \end{pmatrix}.
$$

Other choices of $\boldsymbol{a} = (a_1, \dots, a_n)^{\mathrm{T}} \in \mathbb{R}^n$ and $\alpha > 0$ satisfying (5) yield a wide range of dependency structures. In all these cases the correlation structure of the sample autocovariance function is the same, depending only on the time design matrix $D(h)$ and the matrix M, but not on the vector $\boldsymbol{a}$.

3. The correlation structure

From now on, we consider data on a support of n points, regularly spaced, and having a covariance matrix in the family $\mathscr{S}$. Definitions (1) and (2) of the classical estimator of the autocovariance function give the expression of the time design matrix $D(h)$, of size $n \times n$:

$$
D(h) = \tfrac{1}{2}(P(h) + P(h)^{\mathrm{T}}), \quad 0 \leqslant h \leqslant n-1,
$$

where $P(h)$ is an $n \times n$ matrix with ones on the hth upper diagonal and zero elsewhere, $1 \leqslant h \leqslant n-1$ and $P(0) = I_n$. There are three possible forms of the matrix $D(h)$, depending on $h < n/2$, $h = n/2$, or $h > n/2$, where the size of the upper or lower diagonal of ones is $n-h$. In this situation, the matrix $D(h)$ has a particular form, which allows us to compute the trace of the product of such matrices explicitly.

Lemma 1. *Let $D(h)$ be the time design matrix of the data at lag h, of size $n \times n$, and let $M = I_n - (1/n)\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}$. Then, we have*

$$
\operatorname{tr}[MD(h)MD(h)] = \frac{n-h}{2} + \frac{1}{n^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}(h)\right)^2 - \frac{2}{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}[D^2(h)]_{ij}\right),
$$

and for $h_1 < h_2$

$$
\operatorname{tr}[MD(h_1)MD(h_2)] = \frac{1}{n^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}(h_1)\right)\left(\sum_{i=1}^{n}\sum_{j=1}^{n} D_{ij}(h_2)\right) - \frac{2}{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}[D(h_1)D(h_2)]_{ij}\right).
$$

Proof. By direct computation, we have

$$\begin{aligned}
&\mathrm{tr}[MD(h_1)MD(h_2)]\\
&=\mathrm{tr}\left[D(h_1)D(h_2)-\frac{1}{n}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}D(h_1)D(h_2)-\frac{1}{n}D(h_1)\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}D(h_2)+\frac{1}{n^2}\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}D(h_1)\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}D(h_2)\right]\\
&=\mathrm{tr}[D(h_1)D(h_2)]-\frac{2}{n}\mathrm{tr}[\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}D(h_1)D(h_2)]+\frac{1}{n^2}\mathrm{tr}[\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}D(h_1)\mathbf{1}_n\mathbf{1}_n^{\mathrm{T}}D(h_2)]\\
&=0-\frac{2}{n}\mathrm{tr}[\mathbf{1}_n^{\mathrm{T}}D(h_1)D(h_2)\mathbf{1}_n]+\frac{1}{n^2}\mathrm{tr}[(\mathbf{1}_n^{\mathrm{T}}D(h_1)\mathbf{1}_n)(\mathbf{1}_n^{\mathrm{T}}D(h_2)\mathbf{1}_n)]\\
&=\frac{1}{n^2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}D_{ij}(h_1)\right)\left(\sum_{i=1}^{n}\sum_{j=1}^{n}D_{ij}(h_2)\right)-\frac{2}{n}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}[D(h_1)D(h_2)]_{ij}\right),
\end{aligned}$$

where $\mathrm{tr}[D(h_1)D(h_2)]=0$ for $h_1\neq h_2$. The first formula is obtained with $h_1=h_2=h$ and $\mathrm{tr}[D(h)D(h)]=(n-h)/2$. □

Lemma 2. *Let $D(h)$ be the time design matrix of the data at lag h, of size $n\times n$. We have*

$$\sum_{i=1}^{n}\sum_{j=1}^{n}D_{ij}(h)=n-h,$$

and for $h_1\leqslant h_2$

$$\sum_{i=1}^{n}\sum_{j=1}^{n}[D(h_1)D(h_2)]_{ij}=\begin{cases}\frac{1}{4}(4n-4h_2-2h_1) & \text{if } n-h_1-h_2\geqslant 0,\\ \frac{1}{2}(n-h_2) & \text{if } n-h_1-h_2<0.\end{cases}$$

Proof. This is a straightforward consequence of the three possible forms of $D(h)$ given above. □

With Lemmas 1 and 2, we have the following:

Theorem 3. *Let $\boldsymbol{x}$ be a random vector with an elliptically contoured distribution $\mathrm{EC}_n(\boldsymbol{\mu},\Sigma^*,\phi)$, where $\boldsymbol{\mu}=\mu\mathbf{1}_n$, $\Sigma=-2\phi'(0)\Sigma^*\in\mathscr{S}$, and the kurtosis parameter is $\kappa=0$. Suppose that the data vector $\boldsymbol{x}$ has unidimensional and regular support of n points. Then, the sample autocovariance function $\hat{\gamma}(h)=(1/n)\boldsymbol{x}^{\mathrm{T}}MD(h)M\boldsymbol{x}$ satisfies:*

(a) $$\mathrm{E}(\hat{\gamma}(h))=\frac{\alpha(n-h)}{2n^2\phi'(0)},$$

(b) $$\mathrm{Var}(\hat{\gamma}(h))=\begin{cases}\dfrac{\alpha^2}{4n^4\phi'(0)^2}((n-h)(n^2-2h)-2n(n-2h)) & \text{if } h\leqslant\frac{n}{2},\\[2mm] \dfrac{\alpha^2}{4n^4\phi'(0)^2}(n-h)(n^2-2h) & \text{if } h>\frac{n}{2},\end{cases}$$

and for $h_1<h_2$

(c) $$\mathrm{Cov}(\hat{\gamma}(h_1),\hat{\gamma}(h_2))=\begin{cases}-\dfrac{\alpha^2}{2n^4\phi'(0)^2}(h_1(n-h_2)+n(n-h_2-h_1)) & \text{if } n-h_1-h_2\geqslant 0,\\[2mm] -\dfrac{\alpha^2}{2n^4\phi'(0)^2}h_1(n-h_2) & \text{if } n-h_1-h_2<0,\end{cases}$$

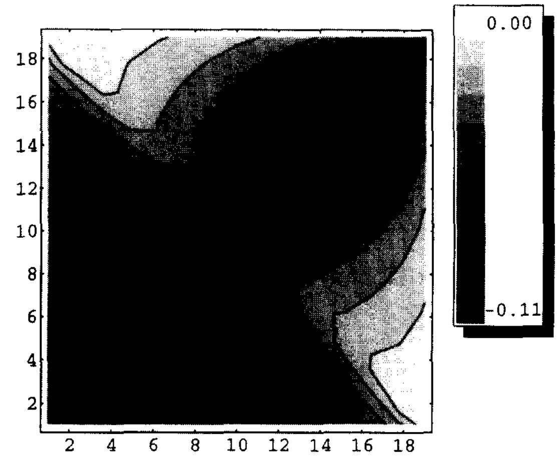

Fig. 1. This plot shows the dependence of $\mathrm{Corr}(\hat{\gamma}(h_1), \hat{\gamma}(h_2))$ on the lags h_1 (horizontal axis) and h_2 (vertical axis), for the sample autocovariance function when the covariance matrix of the elliptically contoured time series is $\Sigma \in \mathscr{S}$. The sample size is $n = 20$. The plot shows the contour lines for correlations between -0.11 and -0.10, between -0.10 and -0.09, etc.

(d) $\mathrm{Corr}(\hat{\gamma}(h_1), \hat{\gamma}(h_2))$

$$
= \begin{cases}
\dfrac{-2(h_1(n-h_2)+n(n-h_2-h_1))}{\sqrt{((n-h_1)(n^2-2h_1)-2n(n-2h_1))((n-h_2)(n^2-2h_2)-2n(n-2h_2))}} & \text{if } h_2 \leqslant \frac{n}{2}, \\[2ex]
\dfrac{-2(h_1(n-h_2)+n(n-h_2-h_1))}{\sqrt{((n-h_1)(n^2-2h_1)-2n(n-2h_1))((n-h_2)(n^2-2h_2))}} & \text{if } h_2 > \frac{n}{2} \text{ and } n-h_1-h_2 \geqslant 0, \\[2ex]
\dfrac{-2h_1(n-h_2)}{\sqrt{((n-h_1)(n^2-2h_1)-2n(n-2h_1))((n-h_2)(n^2-2h_2))}} & \text{if } h_1 \leqslant \frac{n}{2}, \text{ and } n-h_1-h_2 < 0, \\[2ex]
\dfrac{-2h_1(n-h_2)}{\sqrt{((n-h_1)(n^2-2h_1))((n-h_2)(n^2-2h_2))}} & \text{if } h_1 > \frac{n}{2}.
\end{cases}
$$

Note that the covariance in (c) and the correlation in (d) are always negative. These results are similar to those of Dufour and Roy (1985, 1989), and Anderson (1990), but they are valid under much broader conditions, since Theorem 2 holds for all matrices $\Sigma \in \mathscr{S}$. Moreover, the multivariate Gaussian distribution of the time series has been extended to elliptically contoured distributions. A contour plot of the correlation structure of the sample autocovariance function is visualized in Fig. 1, where $n = 20$ and the formula (d) of Theorem 2 is used. The correlation ranges between -0.11 and 0, and can thus be significant for small sample sizes. Its general behavior is similar for other values of n. When the data are dependent through $\Sigma \notin \mathscr{S}$, the correlation between the observations themselves modifies the correlation of the sample autocovariance function.

4. Conclusion

In this paper, the correlation structure of the sample autocovariance function has been derived for specific time series with elliptically contoured distribution. This structure has been studied for a particular family of covariance matrices of the data, thus extending the results in Dufour and Roy (1985, 1989), and Anderson (1990). This special class of time series contains in particular the uncorrelated case and the equicorrelation case. It yields a correlation structure of the sample autocovariance function which does not depend on the underlying dependence structure of the process, but only on the sample size and the lag distance. Therefore, the statistical properties of the sample autocovariance function are the same for all this special class of time series with elliptically contoured distribution.

References

Anderson, O.D., 1982. Sampled correlations from ARIMA processes. Appl. Time Ser. Anal. North-Holland, 5–14.

Anderson, O.D., 1990. Moments of the sampled autocovariances and autocorrelations for a Gaussian white-noise process. The Canad. J. Statist. 18, 271–284.

Anderson, O.D., 1991. All the Gaussian white noise serial covariance moments to order four. Austral. J. Statist. 33, 373–396.

Anderson, O.D., 1993. Exact general-lag serial correlation moments and approximate low-lag partial correlation moments for Gaussian white noise. J. Time Ser. Anal. 14, 551–574.

Anderson, O.D., Chen, Z.-G., 1996. Higher order moments of sample autocovariances and sample autocorrelations from an independent time series. J. Time Ser. Anal. 17, 323–331.

Anderson, O.D., De Gooijer, J.G., 1983. Formulae for the covariance structure of the sample autocovariances from series generated by general autoregressive integrated moving average processes of order (p,d,q), $d=0$ or 1. Sankhyā Ser. B 45, 249–256.

Anderson, O.D., De Gooijer, J.G., 1988. Sampled autocovariance and autocorrelation results for linear time processes. Comm. Statist. Simulation Comput. 17, 489–513.

Box, G.E.P., Jenkins, G.M., 1976. Time Series Analysis: Forecasting and Control. Holden Day, San Francisco, CA.

Brockwell, P.J., Davis, R.A., 1987. Time Series: Theory and Methods. Springer, Berlin.

De Gooijer, J.G., 1980. Exact moments of the sample autocorrelations from series generated by ARIMA processes of order (p,d,q), $d=0$ or 1. J. Econometrics 14, 365–379.

Dufour, J.-M., Roy, R., 1985. Some robust exact results on sample autocorrelations and tests of randomness. J. Econometrics 29, 257–273.

Dufour, J.-M., Roy, R., 1989. Corrigendum: some robust exact results on sample autocorrelations and tests of randomness. J. Econometrics 41, 279–281.

Fang, K.-T., Kotz, S., Ng, K.-W., 1989. Symmetric multivariate and related distributions. Chapman & Hall, London.

Fang, K.-T., Zhang, Y.-T., 1990. Generalized Multivariate Analysis. Springer, Berlin.

Fang, K.-T., Anderson, T.W., 1990. Statistical Inference in Elliptically Contoured and Related Distributions. Allerton Press.

Li, G., 1987. Moments of a random vector and its quadratic forms. J. Statist. Appl. Probab. 2, 219–229.

Li, R.-Z., Fang, K.-T., Zhu, L.-X., 1997. Some Q–Q probability plots to test spherical and elliptical symmetry. J. Comput. Graph. Statist. 6, 435–450.

Muirhead, R.J., 1982. Aspects of Multivariate Statistical Theory. Wiley, New York.