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On fundamental skew distributions

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Abstract

A new class of multivariate skew-normal distributions, fundamental skew-normal distributions and their canonical version, is developed. It contains the product of independent univariate skew-normal distributions as a special case. Stochastic representations and other main properties of the associated distribution theory of linear and quadratic forms are considered. A unified procedure for extending this class to other families of skew distributions such as the fundamental skew-symmetric, fundamental skew-elliptical, and fundamental skew-spherical class of distributions is also discussed.
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1. Introduction

During the last decade, there has been an increasing interest in finding more flexible methods to represent features of the data as adequately as possible and to reduce unrealistic assumptions. The motivation originates from data sets that often do not satisfy some standard assumptions such as independence and normality; see the book edited by Genton [14] for a collection of applications in areas such as economics, finance, oceanography, climatology, environmetrics, engineering, image processing, astronomy, and biomedical sciences.

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One approach for data modeling consists in constructing flexible parametric classes of multivariate distributions that exhibit kurtosis which is different from the normal distribution. The class of elliptical distributions, firstly introduced by Kelker [22] and systematically discussed by Cambanis et al. [12], is probably the most popular example of this approach; see [13] for a comprehensive review. It includes a vast set of known distributions, for example, normal, compound normal, Student-t, power exponential and Pearson type II, among others. Its main advantage is that it represents a natural extension of the concept of symmetry in the multivariate setting. Although elliptical models provide alternatives to the normal model, these can only be applied in practical situations where the symmetry seems reasonable. Therefore, the construction of parametric families of asymmetric distributions which are analytically tractable, can accommodate practical values of skewness and kurtosis, and strictly include the normal distribution, can be useful for data modeling, statistical analysis, and robustness studies of normal theory methods. This work is focused on the study of multivariate skew distributions from a unified approach, and also examines the main properties of such models.

Although the idea of modeling skewness by means of the construction of a mathematically tractable family including the normal distribution was proposed early by other authors (see, for example, [25]), the formal definition of the univariate skew-normal (*SN*, hereafter) family is due to Azzalini [5]. He said that a random variable Z has an *SN* distribution with skewness parameter λ, which is denoted by $Z \sim SN(\lambda)$, if its density is

$$f(z|\lambda) = 2\,\phi(z)\Phi(\lambda z), \quad z \in \mathbb{R}, \quad \lambda \in \mathbb{R}, \tag{1.1}$$

where ϕ and Φ are the $N(0,1)$ probability density function (pdf) and cumulative distribution function (cdf), respectively. The case $\lambda = 0$ reduces (1.1) to the $N(0,1)$ density. Further properties are studied by Azzalini [5,6] and Henze [19]. They show, in particular, that if $Z \sim SN(\lambda)$, then $Z^2 \sim \chi_1^2$ and

$$Z \stackrel{\mathrm{d}}{=} \frac{\lambda}{\sqrt{1+\lambda^2}}|X| + \frac{1}{\sqrt{1+\lambda^2}}Y, \tag{1.2}$$

where X and Y are iid $N(0,1)$ random variables and the notation $X \stackrel{\mathrm{d}}{=} Y$ means that X and Y have the same distribution. Later, Azzalini and Dalla Valle [9] use the stochastic representation (1.2) to extend (1.1) to the multivariate *SN* family of densities, which are given by

$$f(\mathbf{z}|\boldsymbol{\lambda}) = 2\,\phi_k(\mathbf{z})\Phi_1(\boldsymbol{\lambda}^T\mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^k, \quad \boldsymbol{\lambda} \in \mathbb{R}^k, \tag{1.3}$$

where ϕ_k and Φ_k are the pdf and cdf of the k-dimensional normal distribution $N_k(\mathbf{0}, I_k)$, respectively. Again, the case with $\boldsymbol{\lambda} = \mathbf{0}$, reduces (1.3) to the $N_k(\mathbf{0}, I_k)$ density, so that $\boldsymbol{\lambda}$ is interpreted as a shape vector of parameters. Extensions to multivariate location-scale *SN* distributions are also considered in [9]. Further properties of the multivariate *SN* distribution are studied in [7]. Subsequently, Genton et al. [15] derive the moments of random vectors with multivariate *SN* distributions and their quadratic forms (see also [24]).

Generalizations of these ideas have been proposed by many authors. For instance, multivariate distributions such as skew-Cauchy [3], skew-t [8,11,21,26], skew-logistic [27], and other skew-elliptical ones [8,11,4]. Sahu et al. [26] provided a more general way to obtain a family of multivariate skew-elliptical distributions, from which a multivariate version of the univariate *SN* distribution with independent *SN* marginals can be obtained. Recently, Genton and Loperfido [16] introduced a class of generalized skew-spherical (elliptical) distributions defined by densities of the form

$$f(\mathbf{z}|Q) = 2 f_k(\mathbf{z}) Q(\mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^k, \tag{1.4}$$

where f_k is the density corresponding to a k-dimensional spherical (elliptical) distribution (see [13]) and Q is a skewing function, which is such that $Q(\mathbf{z}) \geqslant 0$ and $Q(-\mathbf{z}) = 1 - Q(\mathbf{z})$, for all $\mathbf{z} \in \mathbb{R}^k$. They show that many of the *SN* properties can be extended to any distribution in this class. Strictly speaking, we must note in (1.4) that $Q(\mathbf{z}) = v(u(\mathbf{z}))$, for some function $u : \mathbb{R}^k \to \mathbb{R}$ and some non-negative function $v : \mathbb{R} \to \mathbb{R}$, which are such that $u(-\mathbf{z}) = -u(\mathbf{z})$, for all $\mathbf{z} \in \mathbb{R}^k$, and $v(-u) = 1 - v(u)$, for all $u \in \mathbb{R}$. This alternative representation is used by Azzalini and Capitanio [8] but it is not unique.

More recently, Wang et al. [28] extended (1.4) to any symmetric density f_k in $\mathbb{R}^k$, that is, assuming that f_k satisfies the condition $f_k(-\mathbf{z}) = f_k(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{R}^k$. Further properties and characterizations of these distributions are also discussed in their work. For example, it is shown that under (1.4) the associated distribution theory of linear and quadratic forms remains largely valid. Thus, many multivariate extensions of the univariate *SN* distribution of Azzalini can be obtained as special cases of (1.4), for example, the multivariate *SN* distribution defined in (1.1) by Azzalini and Dalla Valle [9] and that introduced by Azzalini and Capitanio [7]. Moreover, (1.4) generalizes the class of skew-spherical (elliptical) distributions considered in Branco and Dey [11]. However, from (1.4) it is not possible to obtain neither the multivariate family of the skew-spherical (elliptical) distributions considered by Sahu et al. [26], nor the class of multivariate *SN* distributions considered by Gupta et al. [18] and fully discussed by González-Farías et al. [17] (see also [23]), whose canonical version is defined from its density by

$$f(\mathbf{z}|\Phi_m, D) = \frac{\phi_k(\mathbf{z})\Phi_m(D\mathbf{z})}{\Phi_m(\mathbf{0}|I_m + DD^T)}, \quad \mathbf{z} \in \mathbb{R}^k, \tag{1.5}$$

where $\phi_p(\cdot|\boldsymbol{\mu}, \Sigma)$ and $\Phi_p(\cdot|\boldsymbol{\mu}, \Sigma)$ are, respectively, the pdf and cdf of the $N_p(\boldsymbol{\mu}, \Sigma)$ distribution, $\phi_p(\cdot|\Sigma)$ and $\Phi_p(\cdot|\Sigma)$ denote these functions when $\boldsymbol{\mu} = \mathbf{0}$, and D is a matrix of dimension $m \times k$. Note that (1.5) contains also the multivariate *SN* family defined by (1.3) when $m = 1$ and reduces to the product of univariate *SN* distributions when $m = k$ and D is diagonal.

Important additional results are given in Arellano-Valle et al. [2], where a general class of skew-symmetric distributions is introduced starting from the definition of a special $\mathcal{C}$-class of symmetric distributions (or random vectors), defined in terms of independence conditions on signs and absolute values. Specifically, $\mathcal{C}$ is the class of all symmetric random vectors $\mathbf{X}$, with $P(\mathbf{X} = \mathbf{0}) = 0$ and such that: $|\mathbf{X}| = (|X_1|, \ldots, |X_m|)^T$ and $\mathrm{sign}(\mathbf{X}) = (W_1, \ldots, W_m)^T$ are independent, and $\mathrm{sign}(\mathbf{X}) \sim U_m$, where $W_i = +1$, if $X_i > 0$ and $W_i = -1$, if $X_i < 0$, $i = 1, \ldots, m$, and U_m is the uniform distribution on $\{-1, 1\}^m$. Two equivalent stochastic representations, the conditional and marginal representations, are given by Arellano-Valle et al. [2] for any skew random vector obtained from that $\mathcal{C}$-class. Following the results of these authors, we consider here the following general procedure to obtain the density of an arbitrary skew distribution:

$$f(\mathbf{z}|Q_m) = K_m^{-1} f_k(\mathbf{z}) Q_m(\mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^k, \tag{1.6}$$

where

$$K_m = P(\mathbf{X} > \mathbf{0}) \quad \text{and} \quad Q_m(\mathbf{z}) = P(\mathbf{X} > \mathbf{0}|\mathbf{Z} = \mathbf{z}) \tag{1.7}$$

for some random vectors $\mathbf{X}$ and $\mathbf{Z}$ with dimensions $m \times 1$ and $k \times 1$, respectively, and with joint distribution such that $\mathbf{Z}$ has marginal density f_k. Note that (1.6) corresponds to the conditional density of $[\mathbf{Z}|\mathbf{X} > \mathbf{0}]$, which describes a selection model; see Bayarri and De Groot [10] for a survey. As was noted by Arellano-Valle et al. [2], if $\mathbf{X}$ is a $\mathcal{C}$-random vector, then $K_m = P(\mathbf{X} > \mathbf{0}) = 2^{-m}$, so that (1.6) reduces to

$$f(\mathbf{z}|Q_m) = 2^m f_k(\mathbf{z}) Q_m(\mathbf{z}), \quad \mathbf{z} \in \mathbb{R}^k. \tag{1.8}$$

If f_k is a symmetric density on $\mathbb{R}^k$, then Q_m can be interpreted as a skewing function, which in general does not satisfy the condition given by Genton and Loperfido [16] for the family defined by (1.4). In fact, these conditions are satisfied for $m = 1$ only, and in this particular case (1.8) is contained in (1.4). For $m > 1$, (1.8) is more general than (1.4). We note that (1.5) may be obtained from (1.6)–(1.7) by taking $\mathbf{X}|\mathbf{Z} = \mathbf{z} \sim N_m(D\mathbf{z}, I_m)$ and $\mathbf{Z} \sim N_k(\mathbf{0}, I_k)$, so that $\mathbf{X} \sim N_m(\mathbf{0}, I_m + DD^T)$ and $Cov(\mathbf{X}, \mathbf{Z}) = D$. Thus, $f_k(\mathbf{z}) = \phi_k(\mathbf{z})$, $Q_m(\mathbf{z}) = P(\mathbf{X} > \mathbf{0}|\mathbf{Z} = \mathbf{z}) = \Phi_m(D\mathbf{z})$ and $K_m = P(\mathbf{X} > \mathbf{0}) = \Phi_m(\mathbf{0}|I_m + DD^T)$. Note that $K_m = 2^{-m}$ if DD^T is a diagonal matrix. Similarly, if we suppose that $\mathbf{X}|\mathbf{Z} = \mathbf{z} \sim N_m(D^T(\Omega + DD^T)^{-1}\mathbf{z}, I_m - D^T(\Omega + DD^T)^{-1}D)$ and $\mathbf{Z} \sim N_k(\mathbf{0}, \Omega + DD^T)$, so that $\mathbf{X} \sim N_m(\mathbf{0}, I_m)$ and $Cov(\mathbf{Z}, \mathbf{X}) = D$, then from (1.8) we have that

$$\begin{aligned} f(\mathbf{z}|\Phi_m, \Omega, D) = 2^m \phi_k(\mathbf{z}|\Omega + DD^T)\Phi_m(D^T(\Omega + DD^T)^{-1}\mathbf{z}|I_m \\ -D^T(\Omega + DD^T)^{-1}D), \end{aligned} \tag{1.9}$$

which generalizes the Sahu et al. [26] *SN* distribution with null location vector. In fact, these authors consider (1.9) with $m = k$ and assume that Ω and D are diagonal matrices. Note in (1.6)–(1.8) that $Q_m(\mathbf{z}) = v(\mathbf{u}(\mathbf{z}))$, for some function $\mathbf{u} : \mathbb{R}^k \to \mathbb{R}^m$, and some non-negative function $v : \mathbb{R}^m \to \mathbb{R}$.

The main objective of this work is to study the family of skew distributions that follows from (1.6) when in (1.7) the random vector $\mathbf{Z}$ has a particular symmetric distribution. We start by assuming that $\mathbf{Z}$ is normally distributed. Thus, for any conditional distribution of $\mathbf{X}$ given $\mathbf{Z} = \mathbf{z}$, we introduce first in Section 2 the so-called multivariate fundamental skew-normal distributions ("fundamental" because it generalizes all existing definitions of *SN* distributions). Then, we consider a canonical version of that class, which is a special case of the form given in (1.8) with normalizing constant $K_m = 2^{-m}$. We derive for this canonical class several results associated with the distributional theory of linear and quadratic forms, marginals, and conditionals. A marginal stochastic representation for a canonical fundamental skew-normal random vector is also considered. Such representation is used to obtain the moments of these skew-normal random vectors, and may be used also to develop simulation studies related to inferential aspects for these models. Some extensions of these ideas are considered in Section 3, where we show that the general family of skew density defined by (1.6)–(1.7) is closed under marginalization and conditioning. We explore also the special case when the random vector $\mathbf{Z}$ is assumed to be the linear combination $A\mathbf{X} + B\mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated and symmetrically distributed random vectors. Finally, in Section 4 we apply the results obtained in Section 3 to the case where $\mathbf{X}$ and $\mathbf{Y}$ are spherically distributed, thus defining the class of canonical fundamental skew-spherical distributions.

2. Fundamental skew-normal distributions

In this section, we introduce first a very general class of *SN* distributions defined in terms of its density by (1.6)–(1.7), called fundamental skew-normal distributions (*FUSN*). Then, we consider a canonical version of this class and we study its main properties.

Definition 2.1. Let $\mathbf{Z}^* = [\mathbf{Z}|\mathbf{X} > \mathbf{0}]$, where $\mathbf{Z} \sim N_k(\boldsymbol{\mu}, \Sigma)$ and $\mathbf{X}$ is a $m \times 1$ random vector. We say that $\mathbf{Z}^*$ has a *k*-variate fundamental skew-normal (*FUSN*) distribution, which will be denoted by $\mathbf{Z}^* \sim FUSN_{k,m}(\boldsymbol{\mu}, \Sigma, Q_m)$, if its density is given by

$$f_{\mathbf{Z}^*}(\mathbf{z}) = K_m^{-1}\phi_k(\mathbf{z}|\boldsymbol{\mu}, \Sigma)Q_m(\mathbf{z}), \tag{2.1}$$

where $Q_m(\mathbf{z}) = P(\mathbf{X} > \mathbf{0}|\mathbf{Z} = \mathbf{z})$ and $K_m = E[Q_m(\mathbf{Z})] = P(\mathbf{X} > \mathbf{0})$.

Note that K_m is a normalizing constant and that the term Q_m may be interpreted as a skewing function. Thus, as was mentioned in Section 1, from (2.1) we can obtain different families of skew-normal distributions. For instance, if we assume that $\mathbf{X}|\mathbf{Z} = \mathbf{z} \sim N_m(-\boldsymbol{\nu} + D(\mathbf{z} - \boldsymbol{\mu}), \Omega)$, implying that $\mathbf{X} \sim N_m(-\boldsymbol{\nu}, \Omega + D\Sigma D^T)$ and $Cov(\mathbf{Z}, \mathbf{X}) = \Sigma D^T$, then the *SN* distribution introduced by González-Farías et al. [17] follows from (2.1), which reduces to (1.5) when $\boldsymbol{\nu} = \mathbf{0}$ and $\Sigma = \Omega = I_k$. Special cases of (2.1) are obtained when $K_m = 2^{-m}$, which is satisfied by any random vector $\mathbf{X} \in \mathcal{C}$. In this case, we have, for example, a generalization of (1.9) by assuming that $\mathbf{X}|\mathbf{Z} = \mathbf{z} \sim N_m(D^T\Sigma^{-1}_{(\mathbf{z}-\boldsymbol{\mu})}, I_m - D^T\Sigma^{-1}D)$, with $\Sigma = \Omega + DD^T$, so that $\mathbf{X} \sim N_m(\mathbf{0}, I_m)$ and $Cov(\mathbf{Z}, \mathbf{X}) = D$, obtaining from (2.1) that

$$\begin{aligned} f_{\mathbf{Z}^*}(\mathbf{z}) = {} & 2^m\phi_k(\mathbf{z}|\boldsymbol{\mu}, \Omega + DD^T)\Phi_m(D^T(\Omega + DD^T)^{-1}(\mathbf{z} - \boldsymbol{\mu})|I_m \\ & -D^T(\Omega + DD^T)^{-1}D), \end{aligned}$$

which is a location-generalization of (1.9). Note that to obtain the *SN* distribution introduced by Sahu et al. [26], we need to assume $m = k$ and that both matrices, Ω and D, are diagonal.

2.1. The canonical fundamental skew-normal distribution

In this section, we introduce the canonical fundamental skew-normal (*CFUSN*) distribution, for which we consider first the univariate family of *SN* distributions defined by means of its pdf in (1.1). Let $\delta = \lambda/\sqrt{1+\lambda^2}$. Since the relation between $\lambda \in \mathbb{R}$ and $\delta \in (-1, 1)$ is one to one, we have that $\{2\phi(x)\Phi(\lambda x), x \in \mathbb{R}; \lambda \in \mathbb{R}\}$ and $\{2\phi(x)\Phi(\delta x|1 - \delta^2), x \in \mathbb{R}; |\delta| < 1\}$ are equivalent families of *SN*-pdfs. Here, the first parameterization will be denoted by $\{SN(\lambda); \lambda \in \mathbb{R}\}$ and the second by $\{CFUSN(\delta); |\delta| < 1\}$.

Let $\mathbf{Z}^*$ be a $k \times 1$ random vector and let Δ be a $k \times m$ correlation matrix (which is a symmetric matrix when $m = k$) such that $I_m - \Delta^T \Delta$ is a positive definite matrice, i.e., $\|\Delta \mathbf{a}\| < 1$ for any unitary vector $\mathbf{a} \in \mathbb{R}^m$, where $\|\cdot\|$ denotes the norm of a vector.

Definition 2.2. We say that $\mathbf{Z}^*$ has a k-variate canonical fundamental skew-normal (*CFUSN*) distribution with a $k \times m$ skewness matrix Δ, which will be denoted by $\mathbf{Z}^* \sim CFUSN_{k,m}(\Delta)$ and by $\mathbf{Z}^* \sim CFUSN_k(\Delta)$ when $m=k$, if its density is given by

$$f_{\mathbf{Z}^*}(\mathbf{z}) = 2^m \phi_k(\mathbf{z}) \Phi_m(\Delta^T \mathbf{z} | I_m - \Delta^T \Delta), \quad \mathbf{z} \in \mathbb{R}^k, \tag{2.2}$$

where Δ is such that $\|\Delta \mathbf{a}\| < 1$, for all unitary vectors $\mathbf{a} \in \mathbb{R}^m$.

Remark 2.1. Note that the matrix Δ can be constructed as $\Delta = \Lambda(I_m + \Lambda^T \Lambda)^{-1/2}$, for some $k \times m$ real matrix Λ with finite entries. In such case, the matrix identities $I_m - \Delta^T \Delta = (I_m + \Lambda^T \Lambda)^{-1}$ and $I_k - \Delta \Delta^T = (I_k + \Lambda \Lambda^T)^{-1}$ imply that (2.2) can be rewritten as

$$f_{\mathbf{Z}^*}(\mathbf{z}) = 2^m \phi_k(\mathbf{z}) \Phi_m((I_m + \Lambda^T \Lambda)^{-1/2} \Lambda^T \mathbf{z} | (I_m + \Lambda^T \Lambda)^{-1}), \quad \mathbf{z} \in \mathbb{R}^k.$$

Notice the analogy with the univariate *SN* case.

The following lemma will be used frequently.

Lemma 2.1. *Let $\mathbf{Z} \sim N_k(\mathbf{0}, I_k)$. Then, for any fixed vector $\mathbf{u} \in \mathbb{R}^m$ and matrix $A \in \mathbb{R}^{m \times k}$,*

$$E[\phi_m(\mathbf{u} + A\mathbf{Z} | \Omega)] = \phi_m(\mathbf{u} | \Omega + AA^T) \quad \textit{and}$$
$$E[\Phi_m(\mathbf{u} + A\mathbf{Z} | \Omega)] = \Phi_m(\mathbf{u} | \Omega + AA^T).$$

Proof. In fact, $E[\phi_m(\mathbf{u}+A\mathbf{Z}|\Omega)] = \int_{\mathbb{R}^k} \phi_m(\mathbf{u}|-A\mathbf{z}, \Omega)\phi_k(\mathbf{z})\,\mathrm{d}\mathbf{z} = \int_{\mathbb{R}^k} f_{\mathbf{U}|\mathbf{Z}=\mathbf{z}}(\mathbf{u}) f_{\mathbf{Z}}(\mathbf{z})\mathrm{d}\mathbf{z} = f_{\mathbf{U}}(\mathbf{u})$, where $\mathbf{U}|\mathbf{Z} = \mathbf{z} \sim N_m(-A\mathbf{z}, \Omega)$ and $\mathbf{Z} \sim N_k(\mathbf{0}, I_k)$ implying that $\mathbf{U} \sim N_m(\mathbf{0}, \Omega + AA^T)$. $\square$

Note that Lemma 2.1 guarantees in particular that (2.2) is just a density function on $\mathbb{R}^k$, since $\int_{\mathbb{R}^k} \phi_k(\mathbf{z}) \Phi_m(\Delta^T \mathbf{z} | I_m - \Delta^T \Delta)\,\mathrm{d}\mathbf{z} = E[\Phi_m(\Delta^T \mathbf{Z} | I_m - \Delta^T \Delta)] = \Phi_m(\mathbf{0}) = 2^{-m}$.

As it is shown in the next result, the cdf of the *CFUSN* distribution has a simple form.

Proposition 2.1. *If $\mathbf{Z}^* \sim CFUSN_{k,m}(\Delta)$, then its cdf is given by*

$$F_{\mathbf{Z}^*}(\mathbf{z}) = 2^m \Phi_{k+m}((\mathbf{z}^T, \mathbf{0}^T)^T | \Omega), \quad \mathbf{z} \in \mathbb{R}^k \quad \textit{where} \quad \Omega = \begin{pmatrix} I_k & -\Delta \\ -\Delta^T & I_m \end{pmatrix}.$$

Proof. By (2.2),

$$\begin{aligned} F_{\mathbf{Z}^*}(\mathbf{z}) &= 2^m \int_{\mathbf{u} \leqslant \mathbf{0}} \int_{\mathbf{v} \leqslant \mathbf{0}} \phi_k(\mathbf{u} + \mathbf{z}) \phi_m(\mathbf{v} + \Delta^T(\mathbf{u} + \mathbf{z}) | I_m - \Delta^T \Delta)\,\mathrm{d}\mathbf{v}\,\mathrm{d}\mathbf{u} \\ &= 2^m P(\mathbf{U} \leqslant \mathbf{0}, \mathbf{V} \leqslant \mathbf{0}), \end{aligned}$$

where $\mathbf{V}|\mathbf{U}=\mathbf{u}\sim N_m(-\Delta^T(\mathbf{u}+\mathbf{z}), I_m-\Delta^T\Delta)$ and $\mathbf{U}\sim N_k(-\mathbf{z}, I_k)$. Thus, the proof follows from the fact that

$$\begin{pmatrix}\mathbf{U}\\ \mathbf{V}\end{pmatrix}\sim N_{k+m}\left(\begin{pmatrix}-\mathbf{z}\\ \mathbf{0}\end{pmatrix}, \begin{pmatrix}I_k & -\Delta\\ -\Delta^T & I_m\end{pmatrix}\right). \qquad \square$$

2.2. Special cases of the CFUSN distribution

Some important special cases of the *CFUSN* family defined by (2.2) are the following:

1. If $m=1$, with $\Delta=(\delta_1,\ldots,\delta_k)^T$, then (2.2) reduces to the Azzalini and Dalla Valle [9] *SN* density in (1.3), with
$$\boldsymbol{\lambda}=\left(\frac{\delta_1}{\sqrt{1-\sum_{i=1}^k\delta_i^2}},\ldots,\frac{\delta_k}{\sqrt{1-\sum_{i=1}^k\delta_i^2}}\right)^T;$$
2. If $m=k$ and $\Delta=\mathrm{Diag}(\delta_1,\ldots,\delta_k)$, then (2.2) reduces to the product of k univariate *SN* marginals, that is,
$$f_{\mathbf{Z}^*}(\mathbf{z})=\prod_{i=1}^k 2\phi_1(z_i)\Phi_1(\lambda_i z_i)\quad\text{with}\quad \lambda_i=\frac{\delta_i}{\sqrt{1-\delta_i^2}}.$$
Thus, for any univariate *SN* random sample $Z_i^*\sim SN(\lambda)$, $i=1,\ldots,n$, we have that $\mathbf{Z}^*=(Z_1^*,\ldots,Z_n^*)^T\sim CFUSN_n(\delta I_n)$, with $\delta=\lambda/\sqrt{1+\lambda^2}$;
3. If the matrix $\Delta^T\Delta$ is diagonal, i.e., $\Delta_{.i}^T\Delta_{.j}=0$, for all $i\neq j$, where $\Delta_{.j}$, $j=1,\ldots,m$, are the columns of Δ, then (2.2) reduces to
$$f_{\mathbf{Z}^*}(\mathbf{z})=2^m\phi_k(\mathbf{z})\Phi_m(\Lambda^T\mathbf{z})\quad\text{with}\quad \Lambda=\Delta(I_m-\Delta^T\Delta)^{-1/2}.$$
Thus, by letting $D=\Lambda^T=(I_m-\Delta^T\Delta)^{-1/2}\Delta^T$ in (1.5), it follows that a necessary and sufficient condition to obtain (2.2) as special case of (1.5) is that $DD^T=\Lambda^T\Lambda$ be a diagonal matrix. Since this condition is not required by Definition 2.2, then we have in general that (1.5) and (2.2) define different families of *SN* distributions. Note however that (1.5) is a special case of (2.1).

2.3. The CFUSN stochastic representations

The following proposition presents conditional and marginal stochastic representations for the *CFUSN* random vector $\mathbf{Z}^*$ introduced in Definition 2.2.

Proposition 2.2. *Let* $\mathbf{Z}^*\sim CFUSN_{k,m}(\Delta)$, *where* $\|\Delta^T\mathbf{b}\|<1$, *for any unitary vector* $\mathbf{b}\in\mathbb{R}^k$. *Let also* $\mathbf{Z}=\Delta\mathbf{X}+(I_k-\Delta\Delta^T)^{1/2}\mathbf{Y}$, *where* $\mathbf{X}\sim N_m(\mathbf{0},I_m)$ *and* $\mathbf{Y}\sim N_k(\mathbf{0},I_k)$, *which are independent. Then,*

$$\mathbf{Z}^*\overset{\mathrm{d}}{=}[\mathbf{Z}|\mathbf{X}>\mathbf{0}],$$

which is called conditional representation. Moreover,

$$\mathbf{Z}^* \stackrel{\mathrm{d}}{=} \Delta|\mathbf{X}| + (I_k - \Delta\Delta^T)^{1/2}\mathbf{Y} \quad \textit{where} \quad |\mathbf{X}| = (|X_1|, \ldots, |X_m|)^T,$$

which is called marginal representation.

Proof. Theorem 5.1 in Arellano-Valle et al. [2] establishes that the conditional random vector $[\mathbf{Z}|\mathbf{X} > \mathbf{0}]$ has a density as in (1.6), where $f_k(\mathbf{z})$ is the marginal density of $\mathbf{Z}$ and, as is indicated in (1.7), $K_m = P(\mathbf{X} > \mathbf{0})$ and $Q_m(\mathbf{z}) = P(\mathbf{X} > \mathbf{0}|\mathbf{Z} = \mathbf{z})$. Here, by the assumptions it is clear that $\mathbf{Z} \sim N_k(\mathbf{0}, I_k)$, so that the joint distribution of $\mathbf{X}$ and $\mathbf{Z}$ is

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{Z} \end{pmatrix} \sim N_{m+k}\left(\begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \begin{pmatrix} I_m & \Delta^T \\ \Delta & I_k \end{pmatrix}\right)$$

and so $[\mathbf{X}|\mathbf{Z} = \mathbf{z}] \sim N_m(\Delta^T\mathbf{z}, I_m - \Delta^T\Delta)$. Thus, $K_m = 2^{-m}$, $f_k(\mathbf{z}) = \phi_k(\mathbf{z})$ and $Q_m(\mathbf{z}) = \Phi_m(\Delta^T\mathbf{z}|I_m - \Delta^T\Delta)$, from where the conditional representation follows. The marginal representation comes from its equivalence with the conditional representation, which is established in Theorem 3.1 of Arellano-Valle et al. [2]. $\square$

Remark 2.2. Since $\Delta = \Lambda(I_m + \Lambda^T\Lambda)^{-1/2}$, for some real $k \times m$ matrix Λ, it follows from the matrix identity $(I_k + \Lambda\Lambda^T)^{-1} = I_k - \Lambda(I_m + \Lambda^T\Lambda)^{-1}\Lambda^T = I_k - \Delta\Delta^T$ that $\mathbf{Z} = \Lambda(I_m + \Lambda^T\Lambda)^{-1/2}\mathbf{X} + (I_k + \Lambda\Lambda^T)^{-1/2}\mathbf{Y}$. Note also that $Cov(\mathbf{Z}, \mathbf{X}) = \Delta$. Thus, since $V(\mathbf{Z}) = I_k$ and $V(\mathbf{X}) = I_m$, we have for any given $k \times m$ real matrix Λ that $\Delta = \Lambda(I_m + \Lambda^T\Lambda)^{-1/2}$ is a correlation matrix.

Corollary 2.1. *If* $\mathbf{Z}^* \sim CFUSN_{k,m}(\Delta)$, *then*

$$E(\mathbf{Z}^*) = \sqrt{\frac{2}{\pi}}\,\Delta\mathbf{1}_m \quad \text{and} \quad V(\mathbf{Z}^*) = I_k - \frac{2}{\pi}\,\Delta\Delta^T,$$

where $\mathbf{1}_n$ *is a* $n \times 1$ *vector of ones.*

Proof. Using that $E[|\mathbf{X}|] = \sqrt{\frac{2}{\pi}}\mathbf{1}_m$ and $V[|\mathbf{X}|] = (1 - 2/\pi)I_m$, where $\mathbf{X} \sim N_m(\mathbf{0}, I_m)$, the proof is direct from the marginal stochastic representation. $\square$

Note from Corollary 2.1 that if $\mathbf{Z}^* = (Z_1^*, \ldots, Z_k^*)^T \sim CFUSN_{k,m}(\Delta)$, then

$$Cov(Z_i^*, Z_j^*) = -\frac{2}{\pi}\Delta_{i.}^T\Delta_{j.} = -\frac{2}{\pi}\sum_{p=1}^{m}\delta_{ip}\delta_{jp}, \quad i \neq j,$$

where $\Delta_{i.}^T = (\delta_{i1}, \ldots, \delta_{im})$, $i = 1, \ldots, k$, are the rows of Δ. Similarly, if $\mathbf{Z}^*$ and Δ are partitioned in the form of

$$\mathbf{Z}^* = \begin{pmatrix} \mathbf{Z}_1^* \\ \mathbf{Z}_2^* \end{pmatrix} \quad \text{and} \quad \Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}, \tag{2.3}$$

where $\mathbf{Z}_i^*$ and Δ_i have dimensions $k_i \times 1$ and $k_i \times m$, $i = 1, 2$, respectively, and $k_1 + k_2 = k$, then

$$Cov(\mathbf{Z}_1^*, \mathbf{Z}_2^*) = -\frac{2}{\pi}\Delta_1\Delta_2^T.$$

2.4. The CFUSN moment generating function

In the next result, we derive the moment generating function (mgf) of the *CFUSN* distribution. Additional properties of this distribution are obtained from its mgf.

Proposition 2.3. *If $\mathbf{Z}^* \sim CFUSN_{k,m}(\Delta)$, then its mgf is given by*

$$M_{\mathbf{Z}^*}(\mathbf{t}) = 2^m e^{(1/2)\mathbf{t}^T\mathbf{t}}\Phi_m(\Delta^T\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^k.$$

Proof. Note first that $e^{\mathbf{t}^T\mathbf{z}}\phi_k(\mathbf{z}) = e^{(1/2)\mathbf{t}^T\mathbf{t}}\phi_k(\mathbf{z}-\mathbf{t})$. Hence, using the variable change $\mathbf{y} = \mathbf{z} - \mathbf{t}$ after applying (2.2), we have that

$$M_{\mathbf{Z}^*}(\mathbf{t}) = E[e^{\mathbf{t}^T\mathbf{Z}^*}] = 2^m e^{(1/2)\mathbf{t}^T\mathbf{t}}E[\Phi_m(\Delta^T\mathbf{t} + \Delta^T\mathbf{Y}|I_m - \Delta^T\Delta)],$$

where $\mathbf{Y} \sim N_k(\mathbf{0}, I_k)$. Thus, the proof follows from the second equation in Lemma 2.1. □

An important byproduct of Proposition 2.3 is the following additive property of the *CFUSN* distribution.

Corollary 2.2. *Let $\mathbf{Z}_i^* \sim CFUSN_{k,m_i}(\Delta_i)$, $i = 1, \ldots, n$, which are $k \times 1$ independent random vectors. Then,*

$$\bar{\mathbf{Z}}^* = \frac{1}{n}\sum_{i=1}^n \mathbf{Z}_i^* \sim CFUSN_{k,m}(\bar{\Delta}) \quad where \quad m = \sum_{i=1}^n m_i$$
$$and \quad \bar{\Delta} = \frac{1}{n}(\Delta_1, \ldots, \Delta_n).$$

Proof. Since $\prod_{i=1}^n \Phi_{m_i}((1/n)\Delta_i^T\mathbf{t}) = \Phi_m(\bar{\Delta}^T\mathbf{t})$, where $m = \sum_{i=1}^n m_i$ and $\bar{\Delta} = (1/n)$ $(\Delta_1, \ldots, \Delta_n)$, the proof follows from Proposition 2.3. □

Now, we characterize the distribution of an arbitrary linear transformation $A\mathbf{Z}^* + \mathbf{b}$ by means of its mgf and also by means of its pdf when A is a non-singular matrix. The proof of such results is direct from Proposition 2.3 by noting that $M_{A\mathbf{Z}^*+\mathbf{b}}(\mathbf{t}) = e^{\mathbf{t}^T\mathbf{b}}M_{\mathbf{Z}^*}(A^T\mathbf{t})$ and from (2.2) by using that $f_{A\mathbf{Z}^*+\mathbf{b}}(\mathbf{y}) = |\det(A)|^{-1}f_{\mathbf{Z}^*}(A^{-1}(\mathbf{y}-\mathbf{b}))$.

Proposition 2.4. *If $\mathbf{Z}^* \sim CFUSN_{k,m}(\Delta)$, then for any $n \times k$ matrix A,*

$$M_{A\mathbf{Z}^*+\mathbf{b}}(\mathbf{t}) = 2^m e^{\mathbf{t}^T\mathbf{b}+(1/2)\mathbf{t}^TAA^T\mathbf{t}}\Phi_m(\Delta^TA^T\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^k. \tag{2.4}$$

Moreover, if for $n = k$ the matrix A is nonsingular, then

$$f_{A\mathbf{Z}^*+\mathbf{b}}(\mathbf{y}) = 2^m |\det(A)|^{-1} \phi_k(A^{-1}(\mathbf{y}-\mathbf{b}))\Phi_m(\Delta^T A^{-1}(\mathbf{y}-\mathbf{b})|I_m - \Delta^T\Delta), \quad \mathbf{y} \in \mathbb{R}^k. \tag{2.5}$$

By using (2.4) with $\mathbf{b} = \mathbf{0}$, we have the following additional properties of the *CFUSN* distribution.

Corollary 2.3. *Let $\mathbf{Z}^* \sim CFUSN_{k,m}(\Delta)$. Then*
(i) $-\mathbf{Z}^* \sim CFUSN_{k,m}(-\Delta)$;
(ii) $\mathbf{a}^T\mathbf{Z}^* \sim FUSN_{1,m}(\mathbf{a}^T\Delta)$, *for any unitary vector* $\mathbf{a} \in \mathbb{R}^k$;
(iii) $A\mathbf{Z}^* \sim CFUSN_{k,m}(A\Delta)$, *for any $k \times k$ orthogonal matrix A.*

To end this section, we characterize next the distribution of quadratic forms of a *CFUSN* random vector by means of their mgf's.

Proposition 2.5. *If $\mathbf{Z}^* \sim CFUSN_{k,m}(\Delta)$ and A is a given $k \times k$ symmetric matrix, then the mgf of $\Psi = \mathbf{Z}^{*T} A\mathbf{Z}^*$ is*

$$M_\Psi(t) = 2^m |I_k - 2tA|^{-1/2} \Phi_m(\mathbf{0}|I_m + 2t\Delta^T(I_k - 2tA)^{-1}A\Delta), \quad t \in \mathbb{R},$$

which for $A = I_k$ reduces to

$$M_\Psi(t) = 2^m(1-2t)^{-k/2}\Phi_m(\mathbf{0}|I_m + 2t(1-2t)^{-1}\Delta^T\Delta), \quad t \in \mathbb{R}.$$

Proof. Note first that $e^{t\mathbf{z}^T A\mathbf{z}}\phi_k(\mathbf{z}) = |I_k - 2tA|^{-1/2}\phi_k(\mathbf{z}|(I_k - 2tA)^{-1})$. Now, consider the random vector $\mathbf{Z} \sim N_k(\mathbf{0}, (I_k - 2tA)^{-1})$. The proof follows by using (2.2) and the fact that the second equation in Lemma 2.1 implies that $E[\Phi_m(\Delta^T\mathbf{Z}|I_m - \Delta^T\Delta)] = \Phi_m(\mathbf{0}|I_m - \Delta^T\Delta + \Delta^T(I_k - 2tA)^{-1}\Delta)$. $\square$

Note the simplification in Proposition 2.5 when $A\Delta = \mathrm{O}$. From Proposition 2.5, we can obtain the moments of the quadratic form $\Psi = \mathbf{Z}^{*T}A\mathbf{Z}^*$ by differentiating its mgf, following Genton et al. [15]. For instance, from Corollary 2.1,

$$E(\Psi) = \mathrm{tr}(AV(\mathbf{Z}^*)) + E(\mathbf{Z}^{*T})AE(\mathbf{Z}^*) = \mathrm{tr}(A) + \frac{2}{\pi}[\mathbf{1}_m^T\Delta^T A\Delta\mathbf{1}_m - \mathrm{tr}(\Delta^T A\Delta)].$$

Alternatively, since A is a $k \times k$ symmetric matrix with eigenvalues λ_i, $i = 1, \dots, k$, then we can use that $A = P^T\Lambda P$, where $\Lambda = \mathrm{Diag}(\lambda_1, \dots, \lambda_k)$ and P is a $k \times k$ orthogonal matrix, so that

$$\Psi = \mathbf{Z}^{*T}P^T\Lambda P\mathbf{Z}^* \stackrel{\mathrm{d}}{=} \mathbf{Y}^{*T}\Lambda\mathbf{Y}^* = \sum_{i=1}^k \lambda_i Y_i^{*2},$$

where, from part (iii) of Corollary 2.3, $\mathbf{Y}^* = P\mathbf{Z}^* \sim CFUSN_{k,m}(P\Delta)$, and from part (ii) of the same corollary, $Y_i^* = P_{.i}^T\mathbf{Z}^* \sim CFUSN_{1,m}(P_{.i}^T\Delta)$, $i = 1, \dots, k$, where $P_{.1}, \dots, P_{.k}$

are the columns of P (i.e., $P_{.i}$ is the eigenvector of A corresponding to the eigenvalue λ_i, $i = 1, \ldots, k$). Thus, the results given in Section 4 of Arellano-Valle et al. [2] can be used to obtain the moments of Ψ.

Another important byproduct of Proposition 2.5 is the following characterization for the distribution of the squared length of a *CFUSN* random vector.

Corollary 2.4. *If* $\mathbf{Z}^* \sim CFUSN_{k,m}(\Delta)$, *with* $\Delta \neq \mathrm{O}$, *then* $\|\mathbf{Z}^*\|^2 \sim \chi_k^2$ *if and only if the matrix* $\Delta^T\Delta$ *is diagonal*.

2.5. The CFUSN marginal distributions and independence

Denote by $(\mathbf{Z}_i^*, i = 1, 2)$ and by $(\Delta_i, i = 1, 2)$ the partition of the random vector $\mathbf{Z}^*$ and the induced row-partition of the matrix Δ described in (2.3), respectively. Considering this notation, the next result shows that the *CFUSN* distribution is closed under marginalization.

Proposition 2.6. *Assume that* $\mathbf{Z}^* = (\mathbf{Z}_i^*, i = 1, 2) \sim CFUSN_{k,m}(\Delta)$. *Then, we have* $\mathbf{Z}_i^* \sim CFUSN_{k_i,m}(\Delta_i)$, *i.e.*,

$$f_{\mathbf{Z}_i^*}(\mathbf{z}_i) = 2^m \phi_{k_i}(\mathbf{z}_i)\Phi_m(\Delta_i^T\mathbf{z}_i | I_m - \Delta_i^T\Delta_i), \ \mathbf{z}_i \in \mathbb{R}^{k_i}, \quad i = 1, 2. \tag{2.6}$$

Proof. It is direct from (2.4) by taking $A = [I_{k_1}, \mathrm{O}]$ for $i = 1$ and $A = [\mathrm{O}, I_{k_2}]$ for $i = 2$. Alternatively, since $I_m - \Delta^T\Delta = I_m - \Delta_1^T\Delta_1 - \Delta_2^T\Delta_2$, we have from (2.2) that $f_{\mathbf{Z}_1^*}(\mathbf{z}_1) = 2^m\phi_{k_1}(\mathbf{z}_1)E[\Phi_m(\Delta_1^T\mathbf{z}_1 + \Delta_2^T\mathbf{Z}_2 | I_m - \Delta^T\Delta)]$, where $\mathbf{Z}_2 \sim N_{k_2}(\mathbf{0}, I_{k_2})$, which yields (2.6) with $i = 1$ by using the second equation in Lemma 2.1. In a similar way, we obtain (2.6) for $i = 2$. □

Remark 2.3. If $\Delta = \Lambda(I_m + \Lambda\Lambda)^{-1/2}$, then $\Delta_i = \Lambda_i(I_m + \Lambda^T\Lambda)^{-1/2}$, where $(\Lambda_i, i = 1, 2)$ is the corresponding partition on the matrix Λ.

Suppose now that $m > 1$ and let us partition the matrices Δ_i, $i = 1, 2$, indicated in (2.3) as $\Delta_i = (\Delta_{i1}, \Delta_{i2})$, where Δ_{ij} has dimension $k_i \times m_j$, $j = 1, 2$, and $m_1 + m_2 = m$.

Proposition 2.7. *Assume that* $\mathbf{Z}^* = (\mathbf{Z}_i^*, i = 1, 2) \sim CFUSN_{k,m}(\Delta)$, *with* $m > 1$ *and* $\Delta \neq \mathrm{O}$. *Then, under each of the following conditions on the shape matrix* Δ, *the random vectors* $\mathbf{Z}_1^*$ *and* $\mathbf{Z}_2^*$ *are independent*:
(i) $\Delta_{12} = \mathrm{O}$ *and* $\Delta_{21} = \mathrm{O}$. *In this case*, $\mathbf{Z}_i^* \sim CFUSN_{k_i,m_i}(\Delta_{ii})$, $i = 1, 2$; *or*
(ii) $\Delta_{ii} = \mathrm{O}$, $i = 1, 2$. *In this case*, $\mathbf{Z}_1^* \sim CFUSN_{k_1,m_2}(\Delta_{12})$ *and* $\mathbf{Z}_2^* \sim CFUSN_{k_2,m_1}(\Delta_{21})$.

Proof. The proof follows by noting that (2.2) implies

$$\begin{aligned} f_{\mathbf{Z}_1^*,\mathbf{Z}_2^*}(\mathbf{z}_1, \mathbf{z}_2) = {} & 2^{m_1+m_2}\phi_{k_1}(\mathbf{z}_1)\phi_{k_2}(\mathbf{z}_2)\Phi_{m_1+m_2}(\Delta_1^T\mathbf{z}_1 + \Delta_2^T\mathbf{z}_2 | I_{m_1+m_2} \\ & -\Delta_1^T\Delta_1 - \Delta_2^T\Delta_2), \end{aligned} \tag{2.7}$$

where from the partition $\Delta_i = (\Delta_{i1}, \Delta_{i2})$, $i = 1, 2$,

$$\Delta_1^T \mathbf{z}_1 + \Delta_2^T \mathbf{z}_2 = \begin{pmatrix} \Delta_{11}^T \mathbf{z}_1 + \Delta_{21}^T \mathbf{z}_2 \\ \Delta_{12}^T \mathbf{z}_1 + \Delta_{22}^T \mathbf{z}_2 \end{pmatrix}$$

and

$$\begin{aligned} & I_{m_1+m_2} - \Delta_1^T \Delta_1 - \Delta_2^T \Delta_2 \\ & \quad = \begin{pmatrix} I_{m_1} - (\Delta_{11}^T \Delta_{11} + \Delta_{21}^T \Delta_{21}) & -(\Delta_{11}^T \Delta_{12} + \Delta_{21}^T \Delta_{22}) \\ -(\Delta_{12}^T \Delta_{11} + \Delta_{22}^T \Delta_{21}) & I_{m_2} - (\Delta_{12}^T \Delta_{12} + \Delta_{22}^T \Delta_{22}) \end{pmatrix}. \end{aligned}$$

Thus, the last term in (2.7) factorizes into (i) $\Phi_{m_1}(\Delta_{11}^T \mathbf{z}_1 | I_{m_1} - \Delta_{11}^T \Delta_{11}) \Phi_{m_2}(\Delta_{22}^T \mathbf{z}_2 | I_{m_2} - \Delta_{22}^T \Delta_{22})$, if $\Delta_{12} = \mathrm{O}$ and $\Delta_{21} = \mathrm{O}$, and into (ii) $\Phi_{m_2}(\Delta_{12}^T \mathbf{z}_1 | I_{m_2} - \Delta_{12}^T \Delta_{12}) \Phi_{m_1}(\Delta_{21}^T \mathbf{z}_2 | I_{m_1} - \Delta_{21}^T \Delta_{21})$, if $\Delta_{11} = \mathrm{O}$ and $\Delta_{22} = \mathrm{O}$, thus concluding the proof. □

Now, denote by Δ the $(\delta_{ij}; i = 1, \ldots, k, j = 1, \ldots, m)$ and by $\Delta_{i.}^T = (\delta_{i1}, \ldots, \delta_{im})$, $i = 1, \ldots, k$, the rows of the shape matrix Δ. Let also $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)^T$ be the $k \times 1$ vector with 1 at the ith position, $i = 1, \ldots, m$.

Proposition 2.8. *Let* $\mathbf{Z}^* = (Z_1^*, \ldots, Z_k^*)^T \sim CFUSN_{k,m}(\Delta)$. *Then*

(i) $Z_i^* \sim CFUSN_{1,m}(\Delta_{i.}^T)$, $i = 1, \ldots, k$;

(ii) $Z_1^*, \ldots, Z_k^*$ *are independent if and only if* $m = k$ *and* $\Delta_{i.} = \delta_{ij_i} \mathbf{e}_{j_i}$, *for some* $j_i = 1, \ldots, k$, *with* $j_i \neq j_{i'}$ *for all* $i \neq i'$, $i = 1, \ldots, k$. *Moreover, in such case* $Z_i^* \sim CFUSN_1(\delta_{ij_i})$, $i = 1, \ldots, k$.

Proof. Note first that $\mathbf{e}_i^T \mathbf{Z}^* = Z_i^*$ and $\mathbf{e}_i^T \Delta = \Delta_{i.}^T$, where $\|\mathbf{e}_i\| = 1$, $i = 1, \ldots, k$. Thus, by the part (ii) of Corollary 2.3, $M_{Z_i^*}(t_i) = 2^m e^{(1/2)t_i^2} \Phi_m(\Delta_{i.} t_i)$, $i = 1, \ldots, k$, which proves the part (i). To prove (ii), note first that $\prod_{i=1}^k M_{Z_i^*}(t_i) = 2^{km} e^{(1/2) \sum_{i=1}^k t_i^2} \prod_{i=1}^k \Phi_m(\Delta_{i.} t_i)$. Now, by assumption and Proposition 2.3, $M_{Z_1^*, \ldots, Z_k^*}(t_1, \ldots, t_k) = 2^m e^{(1/2) \sum_{i=1}^k t_i^2} \Phi_m(\sum_{i=1}^k \Delta_{i.} t_i)$. Thus, we have that $M_{Z_1^*, \ldots, Z_k^*}(t_1, \ldots, t_k) = \prod_{i=1}^k M_{Z_i^*}(t_i)$, for all $(t_1, \ldots, t_k)$, if and only if $\Phi_m(\sum_{i=1}^k \Delta_{i.} t_i) = 2^{(k-1)m} \prod_{i=1}^k \Phi_m(\Delta_{i.} t_i)$, for all $(t_1, \ldots, t_k)$, which holds if and only if $m = k$ and $\Delta_{i.} = \delta_{ij_i} \mathbf{e}_{j_i}$, for some $j_i = 1, \ldots, k$, with $j_i \neq j_{i'}$ for all $i \neq i'$, $i = 1, \ldots, k$. □

Note that for $m = 1$, it is not possible to obtain independence of all the variables for the components of a *CFUSN* random vector $\mathbf{Z}^*$ unless only one variable is skewed.

2.6. The CFUSN conditional distributions

As is shown in the following result, the *CFUSN* distributions are not closed under conditioning. However, their conditional distributions have the form of the more general family of *FUSN* distributions defined by (2.1).

Proposition 2.9. *If* $\mathbf{Z}^* = (\mathbf{Z}_i^*, i = 1, 2) \sim CFUSN_{k,m}(\Delta)$, *then the conditional density of* $\mathbf{Z}_1^*$ *given* $\mathbf{Z}_2^* = \mathbf{z}_2$ *is*

$$f_{\mathbf{Z}_1^*|\mathbf{Z}_2^*=\mathbf{z}_2}(\mathbf{z}_1) = (K_m(\mathbf{z}_2))^{-1}\phi_{k_1}(\mathbf{z}_1)\Phi_m(\Delta_1^T\mathbf{z}_1| - \Delta_2^T\mathbf{z}_2, I_m - \Delta^T\Delta),$$

where $K_m(\mathbf{z}_2) = \Phi_m(\Delta_2^T\mathbf{z}_2|I_m - \Delta_2^T\Delta_2)$.

Proof. Since $\Phi_m(\Delta^T\mathbf{z}|I_m - \Delta^T\Delta) = \Phi_m(\Delta_1^T\mathbf{z}_1 + \Delta_2^T\mathbf{z}_2|\mathbf{0}, I_m - \Delta^T\Delta) = \Phi_m(\Delta_1^T\mathbf{z}_1| - \Delta_2^T\mathbf{z}_2, I_m - \Delta^T\Delta)$, the proof follows from the fact that (2.6) and (2.7) imply

$$f_{\mathbf{Z}_1^*|\mathbf{Z}_2^*=\mathbf{z}_2}(\mathbf{z}_1) = \frac{\phi_{k_1}(\mathbf{z}_1)\Phi_m(\Delta_1^T\mathbf{z}_1| - \Delta_2^T\mathbf{z}_2, I_m - \Delta^T\Delta)}{\Phi_m(\Delta_2^T\mathbf{z}_2|I_m - \Delta_2^T\Delta_2)}. \qquad \square$$

In order to obtain the conditional moments of $\mathbf{Z}_1^*$ given $\mathbf{Z}_2^* = \mathbf{z}_2$, we need the following preliminary result. Let $\mathbf{Z}_1 \sim N_{k_1}(\mathbf{0}, I_{k_1})$ and let g be a real function such that $E[|g(\mathbf{Z}_1)|] < \infty$. Let also $\mathbf{X}$ be a $m \times 1$ random vector with density $f(\mathbf{x})$ and let

$$f_*(\mathbf{x}|\mathbf{x}_0) = \frac{f(\mathbf{x})}{P(\mathbf{X} \leqslant \mathbf{x}_0)} I_{\{\mathbf{x} \leqslant \mathbf{x}_0\}} \tag{2.8}$$

be the conditional density of $\mathbf{X}$ given $\{\mathbf{X} \leqslant \mathbf{x}_0\}$ for some fixed vector $\mathbf{x}_0 \in \mathbb{R}^m$, where I_A is the indicator of A.

Proposition 2.10. *Let* $\mathbf{Z}^* = (\mathbf{Z}_i^*, i = 1, 2) \sim CFUSN_{k,m}(\Delta)$. *Then, for any integrable real function* g, *it follows that*:

$$E[g(\mathbf{Z}_1^*)|\mathbf{Z}_2^* = \mathbf{z}_2] = E[h(\mathbf{X})|\mathbf{X} \leqslant \Delta_2^T\mathbf{z}_2] \quad with \quad h(\mathbf{X}) = E[g(\mathbf{Z}_1)|\mathbf{X}]$$

where $[\mathbf{Z}_1|\mathbf{X} = \mathbf{x}] \sim N_{k_1}(-\Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}\mathbf{x}, I_{k_1} - \Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}\Delta_1^T)$ *and* $\mathbf{X} \sim N_m(\mathbf{0}, I_m - \Delta_2^T\Delta_2)$.

Proof. Using that g is an integrable real function and considering the fact that $\Phi_m(\Delta_1^T\mathbf{z}_1| - \Delta_2^T\mathbf{z}_2, I_m - \Delta^T\Delta) = \Phi_m(\Delta_2^T\mathbf{z}_2| - \Delta_1^T\mathbf{z}_1, I_m - \Delta^T\Delta)$, we have from Proposition 2.9 that

$$\begin{aligned}
&E[g(\mathbf{Z}_1^*)|\mathbf{Z}_2^* = \mathbf{z}_2] \\
&\quad = \frac{1}{\Phi_m(\Delta_2^T\mathbf{z}_2|I_m - \Delta_2^T\Delta_2)} \int_{\mathbb{R}^{k_1}} g(\mathbf{z}_1)\phi_{k_1}(\mathbf{z}_1)\Phi_m(\Delta_2^T\mathbf{z}_2| - \Delta_1^T\mathbf{z}_1, I_m - \Delta^T\Delta)\,\mathrm{d}\mathbf{z}_1 \\
&\quad = \frac{1}{\Phi_m(\Delta_2^T\mathbf{z}_2|I_m - \Delta_2^T\Delta_2)} \int_{\mathbb{R}^{k_1}} g(\mathbf{z}_1)\phi_{k_1}(\mathbf{z}_1) \\
&\qquad \times \int_{\mathbf{x} \leqslant \Delta_2^T\mathbf{z}_2} \phi_m(\mathbf{x}| - \Delta_1^T\mathbf{z}_1, I_m - \Delta^T\Delta)\,\mathrm{d}\mathbf{x}\,\mathrm{d}\mathbf{z}_1
\end{aligned}$$

$$= \frac{1}{\Phi_m(\Delta_2^T \mathbf{z}_2 | I_m - \Delta_2^T \Delta_2)}$$
$$\times \int_{\mathbf{x} \leqslant \Delta_2^T \mathbf{z}_2} \int_{\mathbb{R}^{k_1}} g(\mathbf{z}_1)\phi_{k_1}(\mathbf{z}_1)\phi_m(\mathbf{x}| - \Delta_1^T \mathbf{z}_1, I_m - \Delta^T \Delta)\, \mathrm{d}\mathbf{z}_1\, \mathrm{d}\mathbf{x}$$
$$= \frac{1}{\Phi_m(\Delta_2^T \mathbf{z}_2 | I_m - \Delta_2^T \Delta_2)} \int_{\mathbf{x} \leqslant \Delta_2^T \mathbf{z}_2} \int_{\mathbb{R}^{k_1}} g(\mathbf{z}_1) f_{\mathbf{Z}_1}(\mathbf{z}_1) f_{\mathbf{X}|\mathbf{Z}_1=\mathbf{z}_1}(\mathbf{x})\, \mathrm{d}\mathbf{z}_1\, \mathrm{d}\mathbf{x},$$

where $f_{\mathbf{Z}_1}(\mathbf{z}_1) = \phi_{k_1}(\mathbf{z}_1)$ and $f_{\mathbf{X}|\mathbf{Z}_1=\mathbf{z}_1}(\mathbf{x}) = \phi_m(\mathbf{x}| - \Delta_1^T \mathbf{z}_1, I_m - \Delta^T \Delta)$, i.e., $\mathbf{Z}_1 \sim N_{k_1}(\mathbf{0}, I_{k_1})$ and $[\mathbf{X}|\mathbf{Z}_1 = \mathbf{z}_1] \sim N_m(-\Delta_1^T \mathbf{z}_1, I_m - \Delta^T \Delta)$. Using that $f_{\mathbf{Z}_1}(\mathbf{z}_1) f_{\mathbf{X}|\mathbf{Z}_1=\mathbf{z}_1}(\mathbf{x}) = f_{\mathbf{X}}(\mathbf{x}) f_{\mathbf{Z}_1|\mathbf{X}=\mathbf{x}}(\mathbf{z}_1)$, where $\mathbf{X} \sim N_m(\mathbf{0}, I_m - \Delta_2^T \Delta_2)$ and so $P(\mathbf{X} \leqslant \Delta_2^T \mathbf{z}_2) = \Phi_m(\Delta_2^T \mathbf{z}_2 | I_m - \Delta_2^T \Delta_2)$, and that $[\mathbf{Z}_1|\mathbf{X} = \mathbf{x}] \sim N_{k_1}(-\Delta_1(I_m - \Delta_2^T \Delta_2)^{-1}\mathbf{x}, I_{k_1} - \Delta_1(I_m - \Delta_2^T \Delta_2)^{-1}\Delta_1^T)$, we have that

$$\begin{aligned} E[g(\mathbf{Z}_1^*)|\mathbf{Z}_2^* = \mathbf{z}_2] &= \frac{1}{\Phi_m(\Delta_2^T \mathbf{z}_2 | I_m - \Delta_2^T \Delta_2)} \\ &\quad \times \int_{\mathbf{x} \leqslant \Delta_2^T \mathbf{z}_2} f_{\mathbf{X}}(\mathbf{x}) \int_{\mathbb{R}^{k_1}} g(\mathbf{z}_1) f_{\mathbf{Z}_1|\mathbf{X}=\mathbf{x}}(\mathbf{z}_1)\, \mathrm{d}\mathbf{z}_1\, \mathrm{d}\mathbf{x}, \\ &= \int_{\mathbb{R}^m} \frac{f_{\mathbf{X}}(\mathbf{x})}{P(\mathbf{X} \leqslant \Delta_2^T \mathbf{z}_2)} I_{\{\mathbf{x} \leqslant \Delta_2^T \mathbf{z}_2\}} \int_{\mathbb{R}^{k_1}} g(\mathbf{z}_1) f_{\mathbf{Z}_1|\mathbf{X}=\mathbf{x}}(\mathbf{z}_1)\, \mathrm{d}\mathbf{z}_1\, \mathrm{d}\mathbf{x}, \\ &= \int_{\mathbb{R}^m} f_*(\mathbf{x}|\Delta_2^T \mathbf{z}_2) E[g(\mathbf{Z}_1)|\mathbf{X} = \mathbf{x}]\, \mathrm{d}\mathbf{x}, \end{aligned}$$

where, as was defined in (2.8), $f_*(\mathbf{x}|\Delta_2^T \mathbf{z}_2)$ is the conditional density of $\mathbf{X}$ given $\{\mathbf{X} \leqslant \Delta_2^T \mathbf{z}_2\}$, which concludes the proof. □

Corollary 2.5. *If* $\mathbf{Z}^* = (\mathbf{Z}_i^*, i = 1, 2) \sim CFUSN_{k,m}(\Delta)$, *then*

$$E[\mathbf{Z}_1^*|\mathbf{Z}_2^* = \mathbf{z}_2] = -\Delta_1(I_m - \Delta_2^T \Delta_2)^{-1} E[\mathbf{X}|\mathbf{X} \leqslant \Delta_2^T \mathbf{z}_2], \tag{2.9}$$

$$\begin{aligned} V[\mathbf{Z}_1^*|\mathbf{Z}_2^* = \mathbf{z}_2] &= I_{k_1} - \Delta_1(I_m - \Delta_2^T \Delta_2)^{-1}\Delta_1^T \\ &\quad + \Delta_1(I_m - \Delta_2^T \Delta_2)^{-1} V[\mathbf{X}|\mathbf{X} \leqslant \Delta_2^T \mathbf{z}_2](I_m - \Delta_2^T \Delta_2)^{-1}\Delta_1^T, \end{aligned} \tag{2.10}$$

where $\mathbf{X} \sim N_m(\mathbf{0}, I_m - \Delta_2^T \Delta_2)$.

Proof. By applying Proposition 2.10 to the function $g(\mathbf{Z}_1^*) = \mathbf{a}^T \mathbf{Z}_1^*$, where $\mathbf{a}$ is an arbitrary vector on $\in \mathbb{R}^{k_1}$, we obtain after using some basic properties of the conditional expectation the following identity:

$$\mathbf{a}^T E[\mathbf{Z}_1^*|\mathbf{Z}_2^* = \mathbf{z}_2] = \mathbf{a}^T E[E(\mathbf{Z}_1|\mathbf{X})|\mathbf{X} \leqslant \Delta_2^T \mathbf{z}_2],$$

which implies that

$$E[\mathbf{Z}_1^*|\mathbf{Z}_2^* = \mathbf{z}_2] = E[E(\mathbf{Z}_1|\mathbf{X})|\mathbf{X} \leqslant \Delta_2^T \mathbf{z}_2].$$

Thus, the proof of (2.9) follows by noting the fact that $[\mathbf{Z}_1|\mathbf{X} = \mathbf{x}] \sim N_{k_1}(-\Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}\mathbf{x}, I_{k_1} - \Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}\Delta_1^T)$ implies $E(\mathbf{Z}_1|\mathbf{X}) = -\Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}\mathbf{X}$, where $\mathbf{X} \sim N_m(\mathbf{0}, I_m - \Delta_2^T\Delta_2)$. Applying again Proposition 2.10 but now to the function $g(\mathbf{Z}_1^*) = (\mathbf{a}^T\mathbf{Z}_1^*)^2$, we have the identity given by

$$\mathbf{a}^T E[\mathbf{Z}_1^*(\mathbf{Z}_1^*)^T|\mathbf{Z}_2^* = \mathbf{z}_2]\mathbf{a} = \mathbf{a}^T E[E(\mathbf{Z}_1\mathbf{Z}_1^T|\mathbf{X})|\mathbf{X} \leqslant \Delta^T\mathbf{z}_2]\mathbf{a}$$

for all $\mathbf{a} \in \mathbb{R}^{k_1}$, that is,

$$E[\mathbf{Z}_1^*(\mathbf{Z}_1^*)^T|\mathbf{Z}_2^* = \mathbf{z}_2] = E[E(\mathbf{Z}_1\mathbf{Z}_1^T|\mathbf{X})|\mathbf{X} \leqslant \Delta^T\mathbf{z}_2].$$

Thus, since

$$\begin{aligned} E[\mathbf{Z}_1\mathbf{Z}_1^T|\mathbf{X}] = \; & I_{k_1} - \Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}\Delta_1^T \\ & + \Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}\mathbf{X}\mathbf{X}^T(I_m - \Delta_2^T\Delta_2)^{-1}\Delta_1^T, \end{aligned}$$

we have that

$$\begin{aligned} E[\mathbf{Z}_1^*(\mathbf{Z}_1^*)^T|\mathbf{Z}_2^* = \mathbf{z}_2] = \; & I_{k_1} - \Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}\Delta_1^T \\ & + \Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}E[\mathbf{X}\mathbf{X}^T|\mathbf{X} \leqslant \Delta^T\mathbf{z}_2](I_m - \Delta_2^T\Delta_2)^{-1}\Delta_1^T \\ = \; & I_{k_1} - \Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}\Delta_1^T \\ & + \Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}V[\mathbf{X}|\mathbf{X} \leqslant \Delta^T\mathbf{z}_2](I_m - \Delta_2^T\Delta_2)^{-1}\Delta_1^T \\ & + \Delta_1(I_m - \Delta_2^T\Delta_2)^{-1}E[\mathbf{X}|\mathbf{X} \leqslant \Delta^T\mathbf{z}_2]E[\mathbf{X}^T|\mathbf{X} \leqslant \Delta^T\mathbf{z}_2] \\ & \times(I_m - \Delta_2^T\Delta_2)^{-1}\Delta_1^T. \end{aligned}$$

and the proof of (2.10) follows by using (2.9). □

Note that in general, in (2.9) and (2.10), it is not easy to obtain closed-form expressions for $E[\mathbf{X}|\mathbf{X} \leqslant \mathbf{x}_0]$ and $V[\mathbf{X}|\mathbf{X} \leqslant \mathbf{x}_0]$ for any given $\mathbf{x}_0$. However, as we show next, for the particular case where $\Delta_2^T\Delta_2 = \mathrm{Diag}(\delta_1^2, \ldots, \delta_m^2)$ we have closed-form solutions for these conditional moments. In fact, since $\mathbf{X} = (X_1, \ldots, X_m)^T \sim N_m(\mathbf{0}, I_m - \Delta_2^T\Delta_2)$, then $X_i \sim N(0, 1 - \delta_i^2)$, $i = 1, \ldots, m$, and are independent, so that

$$E[\mathbf{X}|\mathbf{X} \leqslant \mathbf{x}_0] = (E[X_1|X_1 \leqslant x_{01}], \ldots, E[X_m|X_m \leqslant x_{0m}])^T$$

and

$$V[\mathbf{X}|\mathbf{X} \leqslant \mathbf{x}_0] = \mathrm{Diag}(V[X_1|X_1 \leqslant x_{01}], \ldots, V[X_m|X_m \leqslant x_{0m}]).$$

Now, we may use the following lemma [20].

Lemma 2.2. *If $X \sim N(\mu, \sigma^2)$, then for any given a, it follows that*:

$$\begin{aligned} E[X|X \leqslant a] &= \mu - \frac{\phi(\frac{a-\mu}{\sigma})}{\Phi(\frac{a-\mu}{\sigma})} \quad \textit{and} \\ V[X|X \leqslant a] &= \left\{1 - \frac{\phi(\frac{a-\mu}{\sigma})}{\Phi(\frac{a-\mu}{\sigma})}\left(\frac{a-\mu}{\sigma} + \frac{\phi(\frac{a-\mu}{\sigma})}{\Phi(\frac{a-\mu}{\sigma})}\right)\right\}\sigma^2. \end{aligned}$$

In particular, if $m = 1$, then $X_1 \sim N(0, 1-\delta_1^2)$, so that

$$E[X_1|X_1 \leqslant \Delta_2^T \mathbf{z}_2] = \frac{\phi\left(\frac{\Delta_2^T \mathbf{z}_2}{\sqrt{1-\delta_1^2}}\right)}{\Phi\left(\frac{\Delta_2^T \mathbf{z}_2}{\sqrt{1-\delta_1^2}}\right)}$$

and

$$V[X_1|X_1 \leqslant \Delta_2^T \mathbf{z}_2] = \left\{1 - \frac{\phi\left(\frac{\Delta_2^T \mathbf{z}_2}{\sqrt{1-\delta_1^2}}\right)}{\Phi\left(\frac{\Delta_2^T \mathbf{z}_2}{\sqrt{1-\delta_1^2}}\right)}\left(\frac{\Delta_2^T \mathbf{z}_2}{\sqrt{1-\delta_1^2}} + \frac{\phi\left(\frac{\Delta_2^T \mathbf{z}_2}{\sqrt{1-\delta_1^2}}\right)}{\Phi\left(\frac{\Delta_2^T \mathbf{z}_2}{\sqrt{1-\delta_1^2}}\right)}\right)\right\}(1-\delta_1^2).$$

The above results simplify also when $\Delta_2^T \mathbf{z}_2 = \mathbf{0}$. In fact, let $\mathbf{U} = (I_m - \Delta_2^T \Delta_2)^{-1/2}\mathbf{X} \sim N_m(\mathbf{0}, I_m)$, where $\Delta_2^T \Delta_2$ is a diagonal matrix. Then,

$$\begin{aligned} E[\mathbf{X}|\mathbf{X} \leqslant \mathbf{0}] &= (I_m - \Delta_2^T \Delta_2)^{1/2} E[\mathbf{U}|\mathbf{U} \leqslant \mathbf{0}] \\ &= -(I_m - \Delta_2^T \Delta_2)^{1/2} E[|\mathbf{U}|] \\ &= -\sqrt{\frac{2}{\pi}}(I_m - \Delta_2^T \Delta_2)^{1/2}\mathbf{1}_m \end{aligned}$$

and

$$\begin{aligned} V[\mathbf{X}|\mathbf{X} \leqslant \mathbf{0}] &= (I_m - \Delta_2^T \Delta_2)^{1/2} V[|\mathbf{U}|](I_m - \Delta_2^T \Delta_2)^{1/2} \\ &= (1 - 2/\pi)(I_m - \Delta_2^T \Delta_2). \end{aligned}$$

In such case, it follows from (2.9) and (2.10) that

$$E[\mathbf{Z}_1^*|\mathbf{Z}_2^* = \mathbf{0}] = \sqrt{\frac{2}{\pi}}\Delta_1(I_m - \Delta_2^T \Delta_2)^{-1/2}\mathbf{1}_m \quad \text{and}$$
$$V[\mathbf{Z}_1^*|\mathbf{Z}_2^* = \mathbf{0}] = I_{k_1} - \frac{2}{\pi}\Delta_1(I_m - \Delta_2^T \Delta_2)^{-1}\Delta_1^T.$$

2.7. A location-scale extension of the CFUSN distribution

From the above results, further properties of the *CFUSN* distribution defined by (2.2) can be studied. For instance, if we denote by $CFUSN_{k,m}(\boldsymbol{\mu}, \Sigma; \Delta)$ its corresponding location-scale extension, where $\boldsymbol{\mu}$ is a $k \times 1$ vector and Σ is a $k \times k$ positive definite matrix, then it can easily be introduced by considering the linear transformation $\mathbf{W}^* = \boldsymbol{\mu} + \Sigma^{1/2}\mathbf{Z}^*$, with $\mathbf{Z}^* \sim CFUSN_{k,m}(\Delta)$, whose density and moment generating functions are given by

$$\begin{aligned} f_{\mathbf{W}^*}(\mathbf{w}) &= 2^m |\Sigma|^{-1/2}\phi_k(\Sigma^{-1/2}(\mathbf{w}-\boldsymbol{\mu})) \\ &\quad \times \Phi_m(\Delta^T \Sigma^{-1/2}(\mathbf{w}-\boldsymbol{\mu})|I_m - \Delta^T \Delta), \quad \mathbf{w} \in \mathbb{R}^k, \end{aligned} \tag{2.11}$$

and

$$M_{\mathbf{W}^*}(\mathbf{t}) = 2^m e^{\mathbf{t}^T \boldsymbol{\mu} + (1/2)\mathbf{t}^T \Sigma \mathbf{t}} \Phi_m(\Delta^T \Sigma^{1/2}\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^k,$$

respectively (see (2.5) and (2.4)). In such case, we say $\mathbf{W}^* \sim CFUSN_{k,m}(\boldsymbol{\mu}, \Sigma; \Delta)$. Thus, it follows from Corollary 2.1 that:

$$E(\mathbf{W}^*) = \boldsymbol{\mu} + \sqrt{\frac{2}{\pi}}\, \Sigma^{1/2}\Delta \mathbf{1}_m \quad \text{and} \quad V(\mathbf{W}^*) = \Sigma - \frac{2}{\pi}\, \Sigma^{1/2}\Delta\Delta^T \Sigma^{1/2},$$

where the matrix Δ can be constructed as $\Delta = \Lambda(I_m + \Lambda^T \Lambda)^{-1/2}$ for some real $k \times m$ matrix Λ with finite entries. Note also that a convenient reparametrization is given by $\tilde{\Delta} = \Sigma^{1/2}\Delta$.

An important special case follows when $m = k$, with

$$\Sigma = \mathrm{Diag}(\sigma_1^2, \ldots, \sigma_k^2) \quad \text{and} \quad \Delta = \mathrm{Diag}(\delta_1, \ldots, \delta_k),$$

since under this situation we have that $W_i^* \sim CFUSN_1(\mu_i, \sigma_i^2; \delta_i)$, $i = 1, \ldots, k$, and are independent. Note finally that if in (2.11) we take $\Sigma = \Omega + DD^T$ and $\Delta = (\Omega + DD^T)^{-1/2} D$, then we obtain (1.9), which generalizes the Sahu et al. [26] *SN* distribution.

3. Fundamental skew-symmetric distributions

In this section, we consider the most general class of skew distributions defined in terms of its density by (1.6) and (1.7), called fundamental skew-symmetric distributions (*FUSS*) for each symmetric multivariate distribution with pdf f. We start with the study of some general properties of this class of distributions, considering the following definition based on (1.6) and (1.7).

Definition 3.1. Let $\mathbf{Z}^* = [\mathbf{Z}|\mathbf{X} > \mathbf{0}]$, where $\mathbf{Z}$ is a $k \times 1$ random vector with density f and $\mathbf{X}$ is a $m \times 1$ random vector. If f is a symmetric density on $\mathbb{R}^k$, we say that the random vector $\mathbf{Z}^*$ has a k-variate fundamental skew-symmetric distribution, which will be denoted by $\mathbf{Z}^* \sim FUSS_{k,m}(f, Q_m, K_m)$, if its density is given by

$$f_{\mathbf{Z}^*}(\mathbf{z}) = K_m^{-1} f(\mathbf{z}) Q_m(\mathbf{z}), \tag{3.1}$$

where $Q_m(\mathbf{z}) = P(\mathbf{X} > \mathbf{0}|\mathbf{Z} = \mathbf{z})$ and $K_m = E[Q_m(\mathbf{Z})] = P(\mathbf{X} > \mathbf{0})$.

As in the previous sections, K_m is a normalizing constant and the term Q_m may be interpreted as a skewing function. Thus, as was mentioned in Section 1, from (3.1) we can obtain different families of skew distributions by specifying a symmetric pdf f for $\mathbf{Z}$ and conditional distribution for $\mathbf{X}|\mathbf{Z} = \mathbf{z}$. For instance, if we assume that $\mathbf{Z} \sim N_k(\boldsymbol{\mu}, \Sigma)$, then the *FUSN* class of distributions introduced in (2.1) follows. A class of canonical fundamental skew-symmetric (*CFUSS*) distributions is obtained if we assume that $\mathbf{X} \in \mathcal{C}$, so that $K_m = 2^{-m}$. If $m = 1$, then we obtain the class of skew-symmetric (*SS*) distributions introduced by Wang et al. [28].

3.1. Marginal and conditional distributions of FUSS distributions

In the next result, we characterize the marginal and conditional densities originated by (3.1), for which we consider the partition $\mathbf{Z} = (\mathbf{Z}_i, i = 1, 2)$ and the induced partition on $\mathbf{Z}^* = (\mathbf{Z}_i^*, i = 1, 2)$, where $\mathbf{Z}_i$ and $\mathbf{Z}_i^*$ are $k_i \times 1$ random vectors, $i = 1, 2$, with $k_1 + k_2 = k$.

Proposition 3.1. *Let $\mathbf{Z}^* \sim FUSS_{k,m}(f, Q_m, K_m)$ and let f_i be the marginal density of $\mathbf{Z}_i$, $i = 1, 2$. Then, the marginal density of $\mathbf{Z}_i^*$ is*

$$f_{\mathbf{Z}_i^*}(\mathbf{z}_i) = K_m^{-1} f_i(\mathbf{z}_i) Q_{i,m}(\mathbf{z}_i), \quad with \quad K_m = P(\mathbf{X} > \mathbf{0}) \quad and$$
$$Q_{i,m}(\mathbf{z}_i) = P(\mathbf{X} > \mathbf{0} | \mathbf{Z}_i = \mathbf{z}_i),$$

i.e., $\mathbf{Z}_i^* \sim FUSS_{k_i,m}(f_i, Q_{i,m}, K_m)$, $i = 1, 2$.

Proof. We give the proof for $i = 1$. The proof for $i = 2$ is analogous. By (3.1),

$$\begin{aligned} f_{\mathbf{Z}_1^*}(\mathbf{z}_1) &= K_m^{-1} \int_{\mathbb{R}^{k_2}} f(\mathbf{z}_1, \mathbf{z}_2) Q_m(\mathbf{z}_1, \mathbf{z}_2)\, \mathrm{d}\mathbf{z}_2 \\ &= K_m^{-1} f_1(\mathbf{z}_1) \int_{\mathbb{R}^{k_2}} f_{2|1}(\mathbf{z}_2) Q_m(\mathbf{z}_1, \mathbf{z}_2)\, \mathrm{d}\mathbf{z}_2, \end{aligned}$$

where $f_{i|j}$ is the conditional density of $\mathbf{Z}_i$ given $\mathbf{Z}_j = \mathbf{z}_j$ and, by the properties of the conditional expectation,

$$\begin{aligned} \int_{\mathbb{R}^{k_2}} f_{2|1}(\mathbf{z}_2) Q_m(\mathbf{z}_1, \mathbf{z}_2)\, \mathrm{d}\mathbf{z}_2 &= E(Q_m(\mathbf{Z}_1, \mathbf{Z}_2) | \mathbf{Z}_1 = \mathbf{z}_1) \\ &= E(P(\mathbf{X} > \mathbf{0} | \mathbf{Z}_1 = \mathbf{z}_1, \mathbf{Z}_2) | \mathbf{Z}_1 = \mathbf{z}_1) \\ &= P(\mathbf{X} > \mathbf{0} | \mathbf{Z}_1 = \mathbf{z}_1) \\ &= Q_{1,m}(\mathbf{z}_1). \qquad \square \end{aligned}$$

Corollary 3.1. *The conditional density of $\mathbf{Z}_1^*$ given $\mathbf{Z}_2^* = \mathbf{z}_2$* is

$$f_{\mathbf{Z}_1^* | \mathbf{Z}_2^* = \mathbf{z}_2}(\mathbf{z}_1) = \frac{f(\mathbf{z}_1, \mathbf{z}_2) Q_m(\mathbf{z}_1, \mathbf{z}_2)}{f_2(\mathbf{z}_2) Q_{2,m}(\mathbf{z}_2)} = (K_m(\mathbf{z}_2))^{-1} f_{1|2}(\mathbf{z}_1) Q_m(\mathbf{z}_1 | \mathbf{z}_2),$$

where

$$K_m(\mathbf{z}_2) = Q_{2,m}(\mathbf{z}_2) \quad and \quad Q_m(\mathbf{z}_1 | \mathbf{z}_2) = Q_m(\mathbf{z}_1, \mathbf{z}_2)$$

so that $[\mathbf{Z}_1^* | \mathbf{Z}_2^* = \mathbf{z}_2] \sim FUSS_{k_1,m}(f_{1|2}, Q_m, Q_{2,m})$.

Corollary 3.2. *Consider the partition $\mathbf{X} = (\mathbf{X}_i, i = 1, 2)$, where $\mathbf{X}_i$ is a $m_i \times 1$ random vector, $i = 1, 2$, with $m_1 + m_2 = m > 1$. If $(\mathbf{X}_1, \mathbf{Z}_1)$ and $(\mathbf{X}_2, \mathbf{Z}_2)$ are independent, then $\mathbf{Z}_1^*$ and $\mathbf{Z}_2^*$ are also independent, and their marginal densities are*

$$f_{\mathbf{Z}_i^*}(\mathbf{z}_i) = K_{m_i}^{-1} f_i(\mathbf{z}_i) Q_{m_i}(\mathbf{z}_i),$$

with $Q_{m_i}(\mathbf{z}_i) = P(\mathbf{X}_i > \mathbf{0} | \mathbf{Z}_i = \mathbf{z}_i)$ and $K_{m_i} = E[Q_{m_i}(\mathbf{Z}_i)] = P(\mathbf{X}_i > \mathbf{0})$, $i = 1, 2$.

Proof. In fact, the independence assumption implies $K_m = P(\mathbf{X}_1 > \mathbf{0})P(\mathbf{X}_2 > \mathbf{0}) = K_{m_1}K_{m_2}$, $Q_{1,m}(\mathbf{z}_1) = P(\mathbf{X}_2 > \mathbf{0})P(\mathbf{X}_1 > \mathbf{0}|\mathbf{Z}_1 = \mathbf{z}_1) = K_{m_2}Q_{m_1}(\mathbf{z}_1)$ and, similarly, $Q_{2,m}(\mathbf{z}_2) = K_{m_1}Q_{m_2}(\mathbf{z}_2)$, and $Q_m(\mathbf{z}_1, \mathbf{z}_2) = P(\mathbf{X}_1 > \mathbf{0}|\mathbf{Z}_1 = \mathbf{z}_1)P(\mathbf{X}_2 > \mathbf{0}|\mathbf{Z}_2 = \mathbf{z}_2) = Q_{m_1}(\mathbf{z}_1)Q_{m_2}(\mathbf{z}_2)$. Thus, considering the result in Proposition 3.1, we have the above marginal densities. The independence result is obtained from Corollary 3.1. □

3.2. Obtaining skew distributions from symmetric linear combinations

We consider now the case where $f_{\mathbf{Z}}$ is the density of a linear combination $\mathbf{Z} = A\mathbf{X} + B\mathbf{Y}$, for any given random vector $(\mathbf{X}, \mathbf{Y}) \in \mathbb{R}^{m+n}$ with symmetric density function $f_{\mathbf{X},\mathbf{Y}}$, and where it is assumed (without loss of generality) that $n = k$ and that B is a non-singular matrix. Thus, the skew random vector $\mathbf{Z}^* = [\mathbf{Z}|\mathbf{X} > \mathbf{0}]$ has a density function as in (3.1), which can be obtained in terms of the joint density $f_{\mathbf{X},\mathbf{Y}}$ by using the following result.

Proposition 3.2. *Let $(\mathbf{X}, \mathbf{Y})$ be a random vector with symmetric density $f_{\mathbf{X},\mathbf{Y}}$ and such that $\mathbf{Z} = A\mathbf{X} + B\mathbf{Y}$ has a density $f_{\mathbf{Z}}$. Then, $\mathbf{Z}^* = [\mathbf{Z}|\mathbf{X} > \mathbf{0}]$ has density as in* (3.1) *with*:

$$f_{\mathbf{Z}}(\mathbf{z}) = \frac{1}{|B|}\int_{\mathbb{R}^m} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, B^{-1}\mathbf{z} - B^{-1}A\mathbf{x})\,\mathrm{d}\mathbf{x}$$

and

$$Q_m(\mathbf{z}) = \frac{\int_{\mathbb{R}^m_+} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, B^{-1}\mathbf{z} - B^{-1}A\mathbf{x})\,\mathrm{d}\mathbf{x}}{\int_{\mathbb{R}^m} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, B^{-1}\mathbf{z} - B^{-1}A\mathbf{x})\,\mathrm{d}\mathbf{x}},$$

that is,

$$f_{\mathbf{Z}^*}(\mathbf{z}) = \frac{K_m^{-1}}{|B|}\int_{\mathbb{R}^m_+} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, B^{-1}\mathbf{z} - B^{-1}A\mathbf{x})\,\mathrm{d}\mathbf{x}.$$

Proof. Let $\mathbf{V} = \begin{pmatrix}\mathbf{X}\\ \mathbf{Z}\end{pmatrix}$ and $\mathbf{U} = \begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix}$. Note that

$$\mathbf{V} = \begin{pmatrix}\mathbf{X}\\ A\mathbf{X} + B\mathbf{Y}\end{pmatrix} = \begin{pmatrix} I_m & O\\ A & B\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} = C\mathbf{U},$$

where, since $k = n$ and $|B| \neq 0$, the matrix

$$C = \begin{pmatrix} I_m & O\\ A & B\end{pmatrix} \quad \text{implies} \quad |C| = |B| \quad \text{and} \quad C^{-1} = \begin{pmatrix} I_m & O\\ -B^{-1}A & B^{-1}\end{pmatrix}.$$

Thus,

$$\mathbf{U} = C^{-1}\mathbf{V} = \begin{pmatrix} I_m & O\\ -B^{-1}A & B^{-1}\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{Z}\end{pmatrix} = \begin{pmatrix}\mathbf{X}\\ B^{-1}\mathbf{Z} - B^{-1}A\mathbf{X}\end{pmatrix}.$$

By the Jacobian method, $f_{\mathbf{V}}(\mathbf{v}) = \frac{1}{|C|} f_{\mathbf{U}}(C^{-1}\mathbf{v})$, that is,

$$f_{\mathbf{X},\mathbf{Z}}(\mathbf{x}, \mathbf{z}) = \frac{1}{|B|} f_{\mathbf{X},\mathbf{Y}}(\mathbf{x}, B^{-1}\mathbf{z} - B^{-1}A\mathbf{x})$$

from where the proof follows. $\square$

Note that if we assume the random vectors $\mathbf{X}$ and $\mathbf{Y}$ are uncorrelated with zero mean vector and finite covariance matrices, then we have

$$V(\mathbf{Z}) = AV(\mathbf{X})A^T + BV(\mathbf{Y})B^T, \quad Cov(\mathbf{Z}, \mathbf{X}) = AV(\mathbf{X}) \text{ and}$$
$$Cov(\mathbf{Z}, \mathbf{Y}) = BV(\mathbf{Y}).$$

Hence, if we consider the canonical situation where $V(\mathbf{X}) = I_m$, $V(\mathbf{Y}) = I_n$ and $V(\mathbf{Z}) = I_k$, then we can choose the matrices A and B so that

$$A = Corr(\mathbf{Z}, \mathbf{X}), \quad B = Corr(\mathbf{Z}, \mathbf{Y}) \text{ and } AA^T + BB^T = I_k.$$

Thus, for any given $k \times m$ correlation matrix A, we have $BB^T = I_k - AA^T$. Moreover, since we are assuming that B is a $k \times k$ matrix with $\text{rank}(B) = k$, we can choose $B = (I_k - AA^T)^{1/2}$. Note finally that the matrix A may be constructed as $A = \Lambda(I_m + \Lambda^T\Lambda)^{-1/2}$, so that $B = (I_k + \Lambda\Lambda^T)^{-1/2}$, for any given $k \times m$ matrix Λ. Alternatively, by taking $\Lambda = B^{-1}A$, we have $A = (I_k + \Lambda\Lambda^T)^{-1/2}\Lambda$ and $B = (I_k + \Lambda\Lambda^T)^{-1/2}$.

The assumptions considered above are satisfied, for example, when $(\mathbf{X}, \mathbf{Y})$ is a $\mathcal{C}$-random vector. In such case, as was shown in Arellano-Valle et al. [2], we have that the normalizing constant is $K_m = 2^{-m}$ and we have the following marginal stochastic representation for any skew random vector $\mathbf{Z}^*$ considered in the previous proposition.

Proposition 3.3. *Let* $\mathbf{Z}^* = [\mathbf{Z}|\mathbf{X} > \mathbf{0}]$, *where* $\mathbf{Z} = A\mathbf{X} + B\mathbf{Y}$, *with* $(\mathbf{X}, \mathbf{Y}) \in \mathcal{C}$. *Then*,

$$\mathbf{Z}^* \stackrel{\text{d}}{=} A|\mathbf{X}| + B\mathbf{Y}.$$

Moreover, if we assume the existence of the necessary moments, then

$$E(\mathbf{Z}^*) = AE(|\mathbf{X}|) \quad \text{and} \quad V(\mathbf{Z}^*) = AV(|\mathbf{X}|)A^T + BV(\mathbf{Y})B^T.$$

We can use here the notation $\mathbf{Z}^* \sim FUSS_{k,m}(f_{\mathbf{Z}}, \Delta, K_m)$, where $f_{\mathbf{Z}}$ is the density of the symmetric random vector $\mathbf{Z} = A\mathbf{X} + B\mathbf{Y}$ and Δ is a skewness $k \times m$ matrix depending on A and B and such that $\|\Delta\mathbf{a}\| < 1$, for any unitary $k \times 1$ vector $\mathbf{a}$.

4. The canonical fundamental skew-spherical distribution

In this section, we consider the special case where the density of the skew random vector $\mathbf{Z}^*$ is obtained from the density of a symmetric $\mathcal{C}$-random vector $(\mathbf{X}, \mathbf{Y})$, which is spherically distributed. That is, we assume that

$$\mathbf{U} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \sim S_{m+n}(h^{m+n})$$

for some density generator h^{m+n}. Thus, $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = h^{m+n}(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2)$. Let $\mathbf{Z} = A\mathbf{X} + B\mathbf{Y}$ and let $M = AA^T + BB^T$. Assume also that $\text{rank}(M) = k$. Then, the properties of spherical (elliptical) distributions (see [13]) imply $\mathbf{Z} \sim El_k(\mathbf{0}, M; h^k)$ and has density

$$f_{\mathbf{Z}}(\mathbf{z}) = |M|^{-1/2} h^k(q(\mathbf{z})) \quad \text{with} \quad q(\mathbf{z}) = \mathbf{z}^T M^{-1}\mathbf{z},$$

where

$$h^k(u) = \int_0^\infty \frac{\pi^{(m+n-k)/2}}{\Gamma((m+n)/2)} v^{(m+n-k)/2-1} h^{m+n}(u+v)\,\mathrm{d}v$$

is a marginal generator. Moreover,

$$\mathbf{V} = \begin{pmatrix} \mathbf{X} \\ \mathbf{Z} \end{pmatrix} = C\mathbf{U} \sim El_{m+k}(\mathbf{0}, CC^T; h^{m+k}), \quad \text{where} \quad CC^T = \begin{pmatrix} I_m & A^T \\ A & M \end{pmatrix},$$

which implies

$$[\mathbf{X}|\mathbf{Z} = \mathbf{z}] \sim El_m(\Delta^T M^{-1/2}\mathbf{z}, I_m - \Delta^T\Delta; h^m_{q(\mathbf{z})}),$$

whose conditional density is

$$\begin{aligned} f_{\mathbf{X}|\mathbf{Z}=\mathbf{z}}(\mathbf{x}) = {} & |I_m - \Delta^T\Delta|^{-1/2} h^m_{q(\mathbf{z})}((\mathbf{x} - \Delta^T M^{-1/2}\mathbf{z})^T \\ & \times (I_m - A^T M^{-1} A)^{-1}(\mathbf{x} - \Delta^T M^{-1/2}\mathbf{z})), \end{aligned}$$

where $\Delta = M^{-1/2}A$, $q(\mathbf{z}) = \mathbf{z}^T M^{-1}\mathbf{z}$, and

$$h^m_a(u) = \frac{h^{m+k}(u+a)}{h^k(a)} \tag{4.1}$$

is the conditional generator. Thus, considering the canonical form with $M = I_k$, we get

$$[\mathbf{X}|\mathbf{Z} = \mathbf{z}] \sim El_m(\Delta^T\mathbf{z}, I_m - \Delta^T\Delta; h^k_{q(\mathbf{z})}),$$

where $q(\mathbf{z}) = \|\mathbf{z}\|^2$ and $\mathbf{Z} \sim S_k(h^k)$. Now, denote by $H^m_{q(\mathbf{z})}(\mathbf{x}|I_m - \Delta^T\Delta)$, $\mathbf{x} \in \mathbb{R}^m$, the cdf of $[\mathbf{X} - \Delta^T\mathbf{Z}|\mathbf{Z} = \mathbf{z}] \sim El_m(\mathbf{0}, I_m - \Delta^T\Delta; h^m_{q(\mathbf{z})})$ and consider the random vector defined by $\mathbf{Z}^* = [\mathbf{Z}|\mathbf{X} > \mathbf{0}]$. Note that $Q_m(\mathbf{z}) = P(\mathbf{X} > \mathbf{0}|\mathbf{Z} = \mathbf{z}) = H^m_{q(\mathbf{z})}(\Delta^T\mathbf{z}|I_m - \Delta^T\Delta)$ and $K_m = P(\mathbf{X} > \mathbf{0}) = 2^{-m}$, since $\mathbf{X} \sim S_m(h^m)$. Thus, considering this result, it follows from (3.1) that

$$f_{\mathbf{Z}^*}(\mathbf{z}) = 2^m h^k(q(\mathbf{z})) H^m_{q(\mathbf{z})}(\Delta^T\mathbf{z}|I_m - \Delta^T\Delta) \quad \text{where} \quad q(\mathbf{z}) = \|\mathbf{z}\|^2. \tag{4.2}$$

In such case, we will say that $\mathbf{Z}^*$ has a canonical fundamental skew-spherical distribution with generator h and $k \times m$ shape matrix Δ, which will be denoted by $\mathbf{Z}^* \sim CFUSSPH_{k,m}(\Delta, h^{m+k})$.

Remark 4.1. Note from (4.1) that, for any $a > 0$,

$$H^m_a(\mathbf{x}|\Omega) = \frac{1}{h^k(a)} |\Omega|^{-1/2} \int_{\mathbf{v} \leqslant \mathbf{x}} h^{m+k}(q(\mathbf{v}) + a)\,\mathrm{d}\mathbf{v}, \quad \text{where} \quad q(\mathbf{v}) = \mathbf{v}^T \Omega^{-1}\mathbf{v},$$

so that (4.2) can be rewritten as

$$f_{\mathbf{Z}^*}(\mathbf{z}) = 2^m |I_m - \Delta^T\Delta|^{-1/2} \int_{\mathbf{v}\leqslant \Delta^T\mathbf{z}} h^{m+k}(\mathbf{v}^T(I_m - \Delta^T\Delta)^{-1}\mathbf{v} + \|\mathbf{z}\|^2)\,\mathrm{d}\mathbf{v},$$

which reduces to

$$f_{\mathbf{Z}^*}(\mathbf{z}) = 2^m \int_{\mathbf{u}\leqslant (I_m-\Delta^T\Delta)^{-1/2}\Delta^T\mathbf{z}} h^{m+k}(\|\mathbf{u}\|^2 + \|\mathbf{z}\|^2)\,\mathrm{d}\mathbf{u}$$

when the matrix $\Delta^T\Delta$ is diagonal.

Example 4.1. Let $h^N(u) = (2\pi)^{-N/2}\exp\{-u/2\}$ be the N-dimensional normal generator, where $N = m+k$. Then, $h^k(u) = (2\pi)^{-k/2}\exp\{-u/2\}$ and $h_a^m(u) = h^m(u)$ for all $a > 0$. Thus (4.2) is reduced to (2.2), i.e., $\mathbf{Z}^* \sim CFUSN_{k,m}(\Delta)$.

Example 4.2. Let $h^N(u) = c(N,\nu)\alpha^{\nu/2}\{\alpha+u\}^{-(N+\nu)/2}$, with $c(N,\nu) = \frac{\Gamma((N+\nu)/2)}{\Gamma(\nu/2)}\pi^{N/2}$, be the generator of an N-dimensional (generalized) Student-*t* distribution (see [1]), where ν are the degrees of freedom and α is a scale parameter. Denote this distribution by $t_N(\alpha,\nu)$, and by $t_N(\mu,\Sigma;\alpha,\nu)$ its respective location-scale extension, where $N = m+k$. Then, $h^k(u) = c(k,\nu)\alpha^{\nu/2}\{\alpha+u\}^{-(k+\nu)/2}$ and $h_a^m(u) = c(m,\nu_{(m)})\alpha_a^{\nu_{(m)}/2}\{\alpha_a+u\}^{-(m+\nu_{(m)})/2}$, where $\nu_{(m)} = \nu+N-m = \nu+k$ and $\alpha_a = \alpha+a$, $a>0$, Hence, from (4.2) we have a canonical fundamental skew-*t* distribution defined by the following density:

$$f_{\mathbf{Z}^*}(\mathbf{z}) = 2^m c(k,\nu)\alpha^{\nu/2}\{\alpha+q(\mathbf{z})\}^{-(k+\nu)/2} T_m(\Delta^T\mathbf{z}|I_m - \Delta^T\Delta;\alpha+q(\mathbf{z}),\nu+k),$$

where $q(\mathbf{z}) = \|\mathbf{z}\|^2$ and $T_m(\cdot|I_m-\Delta^T\Delta;\alpha+a,\nu+k)$ denotes the cdf of a $t_m(\mathbf{0}, I_m - \Delta^T\Delta;\alpha+a,\nu+k)$ distribution. We denote this canonical fundamental skew-*t* distribution by $\mathbf{Z}^* \sim CFUST_{k,m}(\Delta,\alpha,\nu)$.

The next result extends the normal marginal stochastic representation given in Proposition 2.2. Its proof is analogous to the normal case, which is a direct consequence of Proposition 3.3, since this proposition establishes that under the $\mathcal{C}$-class, the marginal and conditional stochastic representations of $\mathbf{Z}^*$ are equivalent.

Proposition 4.1. *Let* $\mathbf{Z}^* \sim CFUSSPH_{k,m}(\Delta, h^{m+k})$ *the fundamental skew-spherical distribution defined by* (4.2), *and let* $\mathbf{Z} = \Delta\mathbf{X} + (I_k - \Delta\Delta^T)^{1/2}\mathbf{Y}$, *where* $\begin{pmatrix}\mathbf{X}\\ \mathbf{Y}\end{pmatrix} \sim S_{m+k}(h^{m+k})$. *Then,*

$$\mathbf{Z}^* \overset{\mathrm{d}}{=} [\mathbf{Z}|\mathbf{X} > \mathbf{0}] \overset{\mathrm{d}}{=} \Delta|\mathbf{X}| + (I_k - \Delta\Delta^T)^{1/2}\mathbf{Y}.$$

Moreover, using that for any spherical random vector $\mathbf{W} = (W_1,\ldots,W_N)^T \sim S_N(h)$ *such that* $E[|W_1|^2] < \infty$, *it follows that* $E[\mathbf{W}] = \mathbf{0}$, $V(\mathbf{W}) = \tau_{2,h}I_N$,

$$E[|\mathbf{W}|] = \tau_{1,h}\mathbf{1}_N \quad \text{and} \quad V[|\mathbf{W}|] = \tau_{2,h}(1-2/\pi)I_N + ((2/\pi)\tau_{2,h} - \tau_{1,h}^2)\mathbf{1}_N\mathbf{1}_N^T,$$

where $\tau_{r,h} = E(|W_1|^r)$, $r = 1,2$, ($\tau_{1,h} = \sqrt{(2/\pi)}$ *and* $\tau_{2,h} = 1$ *for the normal distribution*), *then*

$$E(\mathbf{Z}^*) = \tau_{1,h}\Delta\mathbf{1}_m \quad \text{and}$$
$$V(\mathbf{Z}^*) = \tau_{2,h}(I_k - (2/\pi)\Delta\Delta^T) + ((2/\pi)\tau_{2,h} - \tau_{1,h}^2)\Delta\mathbf{1}_m\mathbf{1}_m^T\Delta^T.$$

As in the particular normal case, the extension for a canonical fundamental skew-elliptical distribution $CFUSEl_{k,m}(\boldsymbol{\mu}, \Sigma; \Delta, h^{m+k})$ can be obtained by considering the random vector $\mathbf{W}^* = \boldsymbol{\mu} + \Sigma^{1/2}\mathbf{Z}^*$, with $\mathbf{Z}^* \sim CFSSPH_{k,m}(\Delta, h^{m+k})$, whose density is

$$f_{\mathbf{W}^*}(\mathbf{w}) = 2^m|\Sigma|^{-1/2}h^k(q(\mathbf{w}))H^m_{q(\mathbf{w})}(\Delta^T\Sigma^{-1/2}(\mathbf{w}-\boldsymbol{\mu})|I_m - \Delta^T\Delta),$$

where $q(\mathbf{w}) = (\mathbf{w}-\boldsymbol{\mu})^T\Sigma^{-1}(\mathbf{w}-\boldsymbol{\mu})$.

Another special case of the $\mathcal{C}$-class, is obtained by assuming that $\mathbf{X}\sim S_m(h_1^m)$ and $\mathbf{Y}\sim S_n(h_2^n)$ and are independent. Thus, $f_{\mathbf{X},\mathbf{Y}}(\mathbf{x},\mathbf{y}) = h_1^m(\|\mathbf{x}\|^2)h_2^n(\|\mathbf{y}\|^2)$, so that, by Proposition 3.2,

$$f_{\mathbf{Z}^*}(\mathbf{z}) = \frac{2^m}{|B|}\int_{\mathbb{R}^m_+} h_1^m(\|\mathbf{x}\|^2)h_2^n(\|B^{-1}\mathbf{z} - B^{-1}A\mathbf{x}\|^2)\,\mathrm{d}\mathbf{x}.$$

This case is in general more complicated. However, if h_1^m and h_2^n are normal generators, then this case yields also the *CFUSN* distribution considered in Section 2.

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