Available online at www.sciencedirect.com
ScienceDirect
Journal of Multivariate Analysis 98 (2007) 1876 – 1894
Journal of Multivariate Analysis
www.elsevier.com/locate/jmva

On the exact distribution of linear combinations of order statistics from dependent random variables

Reinaldo B. Arellano-Valle[a], Marc G. Genton[b, c, *]

[a] Departamento de Estadística, Facultad de Matemática, Pontificia Universidad Católica de Chile, Santiago 22, Chile
[b] Department of Econometrics, University of Geneva, Bd du Pont-d'Arve 40, CH-1211 Geneva 4, Switzerland
[c] Department of Statistics, Texas A&M University, College Station, TX 77843-3143, USA

Received 16 January 2006
Available online 3 January 2007

Abstract

We study the exact distribution of linear combinations of order statistics of arbitrary (absolutely continuous) dependent random variables. In particular, we examine the case where the random variables have a joint elliptically contoured distribution and the case where the random variables are exchangeable. We investigate also the particular L-statistics that simply yield a set of order statistics, and study their joint distribution. We present the application of our results to genetic selection problems, design of cellular phone receivers, and visual acuity. We give illustrative examples based on the multivariate normal and multivariate Student t distributions.

AMS 1991 subject classification: 62H05; 62H10; 62E15

Keywords: Elliptically contoured; Exchangeable; Fundamental skew distribution; Kurtosis; Maximum; Selection mechanism; Skewness

1. Introduction

Consider a random vector $\mathbf{X} = (X_1, \dots, X_n)^T$ and assume it has a probability density function (pdf) on $\mathbb{R}^n$. Denote by $\mathbf{X}_{(n)} = (X_{(1)}, \dots, X_{(n)})^T$ the vector of order statistics induced by $\mathbf{X}$, where $X_{(1)} \leqslant \cdots \leqslant X_{(n)}$. We are interested in the problem of finding the exact joint p-dimensional

* Corresponding author. Department of Statistics, Texas A&M University, College Station, TX 77843-3143, USA.
E-mail addresses: reivalle@mat.puc.cl (R.B. Arellano-Valle), Marc.Genton@metri.unige.ch, genton@stat.tamu.edu (M.G. Genton).

0047-259X/$ - see front matter © 2007 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmva.2006.12.003

distribution of linear combinations of the vector of order statistics, also called L-statistics:

$$L\mathbf{X}_{(n)}, \tag{1}$$

for a rectangular matrix $L \in \mathbb{R}^{p\times n}$, under an arbitrary distribution of $\mathbf{X}$. The exact distribution of the maximum of dependent random variables, a particular case of (1) with $p = 1$, has been investigated recently by Arellano-Valle and Genton [4]. Some exact results for the distribution of (1) with $p=1$ have been derived by Gupta and Pillai [15] but only in the setting of a multivariate normal distribution for $\mathbf{X}$ and sizes $n = 3$ and 4. Moreover, some of their results are in error and have been corrected by Nagaraja [20]; see also exercise 5.3.2 in David and Nagaraja [10]. In the particular case of independent and identically distributed (i.i.d.) components of $\mathbf{X}$, some exact results exist for uniform and exponential distributions, see David and Nagaraja [10, Section 6.5] and references therein, but otherwise only asymptotic results have been derived. An interesting particular case of (1) with $p=1$ is the range $X_{(n)} - X_{(1)}$, for which a saddlepoint approximation of its pdf has been given by Ma and Robinson [19] when the components of $\mathbf{X}$ are i.i.d.

The exact pdf of $\mathbf{X}_{(n)}$ when $\mathbf{X}$ is an exchangeable multivariate normal random vector, i.e. its covariance matrix is equicorrelated, has been computed by Tong [25, p. 126], who proposed to derive this pdf as a location mixture of the pdf of order statistics from i.i.d. normal random variables. Thus, an integration problem needs to be solved in order to use Tong's result. We provide another exact form of this pdf and give also an extension to arbitrary (absolutely continuous) dependent random variables, not necessarily normal and/or independent. In particular, when the distribution of $\mathbf{X}$ is exchangeable, we show that the exact distribution of $\mathbf{X}_{(n)}$ belongs to the family of fundamental skew (*FUS*) distributions recently introduced by Arellano-Valle and Genton [3]. Specifically, let $\mathbf{X} = (\mathbf{Y}|\mathbf{Z}\geqslant\mathbf{0})$, where $\mathbf{Y} \in \mathbb{R}^n$ is a random vector with probability density function $f_{\mathbf{Y}}$, $\mathbf{Z} \in \mathbb{R}^m$ is a random vector, and the notation $\mathbf{Z}\geqslant\mathbf{0}$ is meant componentwise. Then Arellano-Valle and Genton [3] say that $\mathbf{X}$ has an n-dimensional *FUS* distribution with pdf given by

$$f_{\mathbf{X}}(\mathbf{x}) = K_m^{-1} f_{\mathbf{Y}}(\mathbf{x}) Q_m(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \tag{2}$$

where $Q_m(\mathbf{x}) = P(\mathbf{Z}\geqslant\mathbf{0}|\mathbf{Y}=\mathbf{x})$ and $K_m = \mathrm{E}(Q_m(\mathbf{Y})) = P(\mathbf{Z}\geqslant\mathbf{0})$. Thus, when $f_{\mathbf{Y}}$ is a symmetric pdf (i.e. $f_{\mathbf{Y}}(-\mathbf{y}) = f_{\mathbf{Y}}(\mathbf{y})$ for all $\mathbf{y} \in \mathbb{R}^n$), (2) defines the fundamental skew-symmetric (*FUSS*) class of distributions. Note that K_m is a normalizing constant and the term Q_m may be interpreted as a function causing skewness in the pdf $f_{\mathbf{X}}$. Indeed, we consider $\mathbf{Y}$ conditionally on $\mathbf{Z}\geqslant\mathbf{0}$ and this selection mechanism induces skewness, see the book edited by Genton [12] for further properties and applications of these distributions, and Arellano-Valle et al. [2] for a unified view on skewed distributions resulting from selections.

The main motivation to study the exact distribution of (1) comes from a genetic selection problem in agricultural research, originally considered by Rawlings [23] and Hill [16,17], and more recently by Tong [25, p. 129]. To describe the problem briefly, suppose that an agricultural genetic selection project involves n animals, for example pigs, and the top k performers, $k < n$, are to be selected for breeding. Let $X_1, \ldots, X_n$ be the measurements of a certain biological or physical characteristic of the n animals, such as the body weights or back fats of the pigs. The animals with score $X_{(n-k+1)}, \ldots, X_{(n)}$ are to be selected. If $X_1, \ldots, X_n$ are independent with mean μ, then the common mean of the observations of off-springs of the ith selected animal with score $X_{(i)}$ is $\mathrm{E}(X_{(i)})$, and therefore the expected gain in one generation is $\mathrm{E}(X_{(i)}) - \mu$. However, the assumption of independence is often not satisfied since the animals under selection are usually genetically related. This is the case, for example, when the pigs are from the same family and have the same parents. In this situation, a variance components model is generally assumed by geneticists, which means that $\mathbf{X} = (X_1, \ldots, X_n)^T$ has an exchangeable multivariate normal distribution [25, p. 108] with a common mean μ, a common variance σ^2 and a common correlation coefficient $\rho \in [0, 1)$. In summary, the distribution of $\mathbf{X}$ is assumed to be multivariate normal $N_n(\mu\mathbf{1}_n, \sigma^2\{(1-\rho)I_n + \rho\mathbf{1}_n\mathbf{1}_n^T\})$, with $\rho \in [0, 1)$, $\mathbf{1}_n \in \mathbb{R}^n$ a vector of ones, and $I_n \in \mathbb{R}^{n\times n}$ the identity matrix. Our goal is to derive an explicit form for the joint pdf of $X_{(n-k+1)}, \ldots, X_{(n)}$ in the above context, but also in more general settings of dependence. In addition, the L-statistics $\sum_{i=n-k+1}^{n} X_{(i)}$ are related to selection differentials [10, pp. 41–42] and are of great interest in the agricultural genetic selection problem described above. We derive their exact distribution and study the shape of their pdf as a function of ρ.

A second motivation comes from the design of future cellular phone receivers that will select the strongest $n-k$ signals $X_{(k+1)}, \ldots, X_{(n)}$ out of n received signals $X_1, \ldots, X_n$ from n antennae, and then process these signals. Under the assumption that $X_i \sim N(\mu_i, \sigma^2)$, $i = 1, \ldots, n$, i.i.d., Wiens et al. [27] have derived the exact distribution of the L-statistics $\sum_{i=k+1}^{n} X_{(i)}$. We extend the computation of this exact distribution to elliptically contoured [11] random variables.

Another motivation comes from vision research where a single measure of visual acuity is made in each eye, X_1 and X_2 say. A person's vision total impairment is defined as the L-statistic $TI = \frac{3}{4}X_{(1)} + \frac{1}{4}X_{(2)}$, where the extremes of visual acuity are $X_{(1)} = \min\{X_1, X_2\}$ and $X_{(2)} = \max\{X_1, X_2\}$, see Viana [26] and references therein. A bivariate normal distribution is commonly assumed for (X_1, X_2). We compute the exact distribution of the total impairment *TI* in this case, and explore the effect of relaxing this assumption to more heavy-tailed distributions. The particular case of the distribution of $X_{(i)}$, $i = 1, 2$, for a bivariate normal random vector has been studied by Roberts [24], Cain [6], Cain and Pan [7], Olkin and Viana [21], and Loperfido [18].

Although this paper focuses on the exact distribution of L-statistics, the derivation of closed-form expressions for moments and cumulants of L-statistics is of great interest too, as described in the genetic selection problem above. We believe that the results of this paper are useful for such derivations, and we give a simple example related to visual acuity in Section 5.3. However, general derivations are beyond the scope of this paper and we leave them for future research.

The structure of the paper is set up as follows. In Section 2, we derive our main results, namely the exact pdf of (1) for arbitrary dependent random variables. In particular, we examine the case where the random variables have a joint elliptically contoured distribution. In Section 3, we focus on exchangeable random variables. We present also results for the particular case of contrast L-statistics where the matrix L in (1) satisfies $L\mathbf{1}_n = \mathbf{0}$. In Section 4, we focus on the particular L-statistics that yield p order statistics, and study their joint distribution. Finally, in Section 5, we revisit the applications to genetic selection, design of cellular phone receivers, and visual acuity. We give illustrative examples based on the multivariate normal and multivariate Student t distributions.

2. L-statistics from dependent random variables

2.1. Arbitrary distributions

In this section, we study the distribution of the L-statistics (1) for which the vector of order statistics $\mathbf{X}_{(n)} = (X_{(1)}, \ldots, X_{(n)})^T$ is induced by an arbitrary (absolutely continuous) random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$. First, note that $\mathbf{X}_{(n)} \in \mathcal{P}(\mathbf{X})$, where $\mathcal{P}(\mathbf{X}) = \{\mathbf{X}_i = P_i\mathbf{X};\ i = 1, \ldots, n!\}$ is the collection of random vectors $\mathbf{X}_i$ corresponding to the $n!$ different permutations of the components of $\mathbf{X}$. Here $P_i \in \mathbb{R}^{n\times n}$ are permutation matrices with $P_i \neq P_j$ for all $i \neq j$. Let

$\Delta \in \mathbb{R}^{(n-1)\times n}$ be the difference matrix such that $\Delta \mathbf{X} = (X_2 - X_1, X_3 - X_2, \ldots, X_n - X_{n-1})^T$, i.e., the ith row of Δ is $\mathbf{e}_{n,i+1}^T - \mathbf{e}_{n,i}^T$, $i = 1, \ldots, n-1$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the n-dimensional unit basis vectors. The region

$$\mathcal{S}_p = \{\mathbf{y} = L\mathbf{x};\ \Delta\mathbf{x} \geqslant \mathbf{0},\ \mathbf{x} \in \mathbb{R}^n\} \tag{3}$$

is defined on $\mathbb{R}^p$ and represents the support of any linear selection based on the order of a random vector $\mathbf{X}$, i.e. $L\mathbf{X}|\Delta\mathbf{X} \geqslant \mathbf{0}$. Our most general result is the following.

Proposition 1. *Let $\mathbf{X} = (X_1, \ldots, X_n)^T$ be an absolutely continuous random vector and denote by $\mathbf{X}_i$ its ith permutation and by $\mathbf{X}_{(n)} = (X_{(1)}, \ldots, X_{(n)})^T$ its corresponding vector of order statistics. Let $L \in \mathbb{R}^{p\times n}$ and suppose that for each $i = 1, \ldots, n!$, $L\mathbf{X}_i$ has a pdf on $\mathbb{R}^p$ denoted by $f_{L\mathbf{X}_i}$. Then, the pdf $f_{L\mathbf{X}_{(n)}}$ of $L\mathbf{X}_{(n)}$ is*

$$f_{L\mathbf{X}_{(n)}}(\mathbf{y}) = \sum\nolimits_{i=1}^{n!} f_{L\mathbf{X}_i}(\mathbf{y}) P(\Delta\mathbf{X}_i \geqslant \mathbf{0} | L\mathbf{X}_i = \mathbf{y}), \quad \mathbf{y} \in \mathcal{S}_p. \tag{4}$$

Proof. Denote by $F_{L\mathbf{X}_{(n)}}$ and $F_{L\mathbf{X}_i|\Delta\mathbf{X}_i \geqslant \mathbf{0}}$ the cumulative distribution functions (cdf) of $L\mathbf{X}_{(n)}$ and $L\mathbf{X}_i|\Delta\mathbf{X}_i \geqslant \mathbf{0}$, respectively. By the theorem of total probability, we have for $\mathbf{y} \in \mathcal{S}_p$:

$$F_{L\mathbf{X}_{(n)}}(\mathbf{y}) = \sum_{i=1}^{n!} F_{L\mathbf{X}_i|\Delta\mathbf{X}_i \geqslant \mathbf{0}}(\mathbf{y}) P(\Delta\mathbf{X}_i \geqslant \mathbf{0}),$$

implying, after differentiation with respect to $\mathbf{y}$, that

$$f_{L\mathbf{X}_{(n)}}(\mathbf{y}) = \sum\nolimits_{i=1}^{n!} f_{L\mathbf{X}_i|\Delta\mathbf{X}_i \geqslant \mathbf{0}}(\mathbf{y}) P(\Delta\mathbf{X}_i \geqslant \mathbf{0}).$$

Thus, the proof follows by noting from Bayes' theorem that

$$f_{L\mathbf{X}_i|\Delta\mathbf{X}_i \geqslant \mathbf{0}}(\mathbf{y}) = f_{L\mathbf{X}_i}(\mathbf{y}) \frac{P(\Delta\mathbf{X}_i \geqslant \mathbf{0} | L\mathbf{X}_i = \mathbf{y})}{P(\Delta\mathbf{X}_i \geqslant \mathbf{0})}. \qquad \square$$

In the particular case $L = I_n$, we have $P(\Delta\mathbf{X}_i \geqslant \mathbf{0} | L\mathbf{X}_i = \mathbf{y}) = I\{\Delta\mathbf{y} \geqslant \mathbf{0}\}$, an indicator function, for all i. Note from the last two expressions of the proof above that the resulting pdf (4) is a mixture on the *FUS* pdf (2) corresponding to $L\mathbf{X}_i|\Delta\mathbf{X}_i \geqslant \mathbf{0}$, $i = 1, \ldots, n!$. The computation of this pdf will, however, be laborious in most situations, particularly for large values of n. Nevertheless, there is a number of situations where the computation of (4) simplifies significantly as will be shown in the next sections.

2.2. Elliptically contoured distributions

We consider next the distribution of (1) when the original random vector $\mathbf{X}$ follows an elliptically contoured distribution; see Cambanis et al. [8] for the most general definition of this family. We say that a random vector $\mathbf{X} \in \mathbb{R}^n$ has an elliptically contoured distribution with location vector $\boldsymbol{\mu} \in \mathbb{R}^n$, nonnegative definite dispersion matrix $\Sigma \in \mathbb{R}^{n\times n}$, and characteristic generator φ, if the centered random vector $\mathbf{X} - \boldsymbol{\mu}$ has characteristic function of the form $\varphi_{\mathbf{X}-\boldsymbol{\mu}}(\mathbf{t}) = \varphi(\mathbf{t}^T\Sigma\mathbf{t})$, for $\mathbf{t} \in \mathbb{R}^n$. In such case, we write $\mathbf{X} \sim EC_n(\boldsymbol{\mu}, \Sigma, \varphi)$, and its cdf will be denoted by $F_n(\mathbf{x}; \boldsymbol{\mu}, \Sigma, \varphi)$, for $\mathbf{x} \in \mathbb{R}^n$. It is well known that this family of distributions is closed under linear transformations, marginalization, and conditioning. In particular, if $\mathbf{X} \sim EC_n(\boldsymbol{\mu}, \Sigma, \varphi)$ with Σ of full rank n, then $\Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim EC_n(\mathbf{0}, I_n, \varphi)$. Moreover, if the pdf of $\mathbf{X}$ exists, then it is of the form

$$f_n(\mathbf{x}; \boldsymbol{\mu}, \Sigma, h^{(n)}) = |\Sigma|^{-1/2} h^{(n)}[(\mathbf{x} - \boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})], \quad \mathbf{x} \in \mathbb{R}^n, \tag{5}$$

where $h^{(n)}$ is the density generator function, see also Fang et al. [11]. In this case, φ can be replaced by $h^{(n)}$ in the above notation. Moreover, for each $k = 1, \ldots, n-1$, we have that any n-dimensional density generator $h^{(n)}$ has an associated k-dimensional marginal density generator and an $(n-k)$-dimensional conditional density generator, which can be expressed in terms of $h^{(n)}$ by $h^{(k)}(u) = \{\pi^{(n-k)/2}/\Gamma[(n-k)/2]\} \int_0^\infty u^{(n-k)/2-1} h^{(n)}(u+v)dv$ and by $h_a^{(n-k)}(u) = h^{(n)}(u+a)/h^{(k)}(a)$, with $a, u \geqslant 0$, respectively. That is, for any partition $(\mathbf{X}_1^T, \mathbf{X}_2^T)^T$ of $\mathbf{X}$, where $\mathbf{X}_1 \in \mathbb{R}^k$, the generators $h^{(k)}$ and $h_a^{(n-k)}$ determine the pdfs of $\mathbf{X}_1$ and $\mathbf{X}_2|\mathbf{X}_1 = \mathbf{x}_1$, respectively. The same occurs with the characteristic generators, but in this case it is not possible to express directly the conditional generators φ_a in terms of φ. In this setting, we have the following result.

Proposition 2. *Let* $\mathbf{X} = (X_1, \ldots, X_n)^T \sim EC_n(\boldsymbol{\mu}, \Sigma, \varphi)$ *and suppose that* $\mathbf{X}$ *has a pdf given by* (5). *Then, the pdf* $f_{L\mathbf{X}_{(n)}}$ *of* $L\mathbf{X}_{(n)}$ *is*

$$\begin{aligned} f_{L\mathbf{X}_{(n)}}(\mathbf{y}) = \sum_{i=1}^{n!} & f_p(\mathbf{y}; L\boldsymbol{\mu}_i, L\Sigma_{ii}L^T, h^{(p)}) \\ & \times F_{n-1}(\Delta\Sigma_{ii}L^T(L\Sigma_{ii}L^T)^{-1}(\mathbf{y} - L\boldsymbol{\mu}_i); \\ & \mathbf{0}, \Delta\Sigma_{ii}\Delta^T - \Delta\Sigma_{ii}L^T(L\Sigma_{ii}L^T)^{-1}L\Sigma_{ii}\Delta^T, \varphi_{q_i(\mathbf{y})}), \quad \mathbf{y} \in \mathcal{S}_p, \end{aligned} \tag{6}$$

where $\boldsymbol{\mu}_i = P_i\boldsymbol{\mu}$, $\Sigma_{ii} = P_i\Sigma P_i^T$, $q_i(\mathbf{y}) = (\mathbf{y} - L\boldsymbol{\mu}_i)^T(L\Sigma_{ii}L^T)^{-1}(\mathbf{y} - L\boldsymbol{\mu}_i)$, *and* $P_i \in \mathbb{R}^{n\times n}$ *is the ith permutation matrix.*

Proof. By the properties of elliptically contoured distributions (see [8]),

$$L\mathbf{X}_i = LP_i\mathbf{X} \sim EC_p(L\boldsymbol{\mu}_i, L\Sigma_{ii}L^T, \varphi),$$

with pdf given by

$$f_{L\mathbf{X}_i}(\mathbf{y}) = f_p(\mathbf{y}; L\boldsymbol{\mu}_i, \Sigma_{ii}, h^{(p)}) = |L\Sigma_{ii}L^T|^{-1/2} h^{(p)}(q_i(\mathbf{y})),$$

where $\boldsymbol{\mu}_i = P_i\boldsymbol{\mu}$ and $\Sigma_{ii} = P_i\Sigma P_i^T$ are the location and the dispersion of the ith permutation $\mathbf{X}_i = P_i\mathbf{X}$ of $\mathbf{X}$, and $q_i(\mathbf{y}) = (\mathbf{y} - L\boldsymbol{\mu}_i)^T(L\Sigma_{ii}L^T)^{-1}(\mathbf{y} - L\boldsymbol{\mu}_i)$. Moreover, $\Delta\mathbf{X}_i|L\mathbf{X}_i = \mathbf{y}$ has distribution

$$EC_{n-1}(\Delta\Sigma_{ii}L^T(L\Sigma_{ii}L^T)^{-1}(\mathbf{y} - L\boldsymbol{\mu}_i), \Delta\Sigma_{ii}\Delta^T - \Delta\Sigma_{ii}L^T(L\Sigma_{ii}L^T)^{-1}L\Sigma_{ii}\Delta^T, \varphi_{q_i(\mathbf{y})}),$$

and thus the probability $P(\Delta\mathbf{X}_i \geqslant \mathbf{0}|L\mathbf{X}_i = \mathbf{y})$ is equal to

$$F_{n-1}(\Delta\Sigma_{ii}L^T(L\Sigma_{ii}L^T)^{-1}(\mathbf{y} - L\boldsymbol{\mu}_i); \mathbf{0}, \Delta\Sigma_{ii}\Delta^T - \Delta\Sigma_{ii}L^T(L\Sigma L^T)^{-1}L\Sigma_{ii}\Delta^T, \varphi_{q_i(\mathbf{y})}).$$

Then the result follows from Proposition 1. □

For L-statistics of the form $L\mathbf{X}_{(n)} = \sum_{r=1}^n a_r X_{(r)}$, i.e., with $p = 1$ and $L = \mathbf{a}^T = (a_1, \ldots, a_n)$, we have some relevant simplification when $\Sigma = \sigma^2 I_n$, since $\Sigma_{ii} = \sigma^2 I_n$ for all $i = 1, \ldots, n!$, and thus (6) reduces to

$$f_{L\mathbf{X}_{(n)}}(y) = \sum_{i=1}^{n!} f_1(y; \eta_i, \tau^2, h^{(1)}) F_{n-1}(\Delta\tilde{\mathbf{a}} z_i; \mathbf{0}, \Delta\{I_n - \tilde{\mathbf{a}}\tilde{\mathbf{a}}^T\}\Delta^T, h^{(n-1)}_{z_i^2}), \quad y \in \mathbb{R}, \quad (7)$$

where $z_i = (y - \eta_i)/\tau$, $\eta_i = \sum_{j=1}^n a_j \mu_{i_j}$, $\tau^2 = \sigma^2 \sum_{j=1}^n a_j^2$ and $\tilde{\mathbf{a}} = \mathbf{a}/\sqrt{\sum_{j=1}^n a_j^2}$. Additional simplifications occur for some special cases, such as when the vector $L = \mathbf{a}^T$ satisfies $L\boldsymbol{\mu}_i = \mu_i = 0$ for all $i = 1, \ldots, n!$; or $a_j = 1$ for all $j = 1, \ldots, n$; or $a_1 = \cdots = a_k = 0$ and $a_{k+1} = \cdots = a_n = 1$. In this latter case, the pdf of $L\mathbf{X}_{(n)} = \sum_{i=k+1}^n X_{(i)}$, is given by (7) with $\eta_i = \sum_{j=k+1}^n \mu_{i_j}$, and in (7) the summation $\sum_{i=1}^{n!}$ reduces to $k!(n-k)!\sum_{i=1}^{n_k}$, where $n_k = n!/[k!(n-k)!]$.

For the normal case, we have $\varphi_a(s) = \varphi(s) = \exp\{s/2\}$, $s \geqslant 0$, and $h_a^{(m)}(u) = h^{(m)}(u) = (2\pi)^{-m/2}\exp\{-u/2\}, a, u \geqslant 0$. The notation $f_n(\mathbf{x}; \boldsymbol{\mu}, \Sigma, h^{(n)}) = \phi_n(\mathbf{x}; \boldsymbol{\mu}, \Sigma)$ and $F_n(\mathbf{x}; \boldsymbol{\mu}, \Sigma, \varphi) = \Phi_n(\mathbf{x}; \boldsymbol{\mu}, \Sigma)$ will be used in this case.

When $\mathbf{X} \sim EC_n(\boldsymbol{\mu}, \Sigma, \varphi)$, we have $\Delta\mathbf{X} \sim EC_{n-1}(\Delta\boldsymbol{\mu}, \Delta\Sigma\Delta^T, \varphi)$, hence $P(\Delta\mathbf{X} \geqslant \mathbf{0}) = F_{n-1}(\Delta\boldsymbol{\mu}; \mathbf{0}, \Delta\Sigma\Delta^T, \varphi)$. An important simplification in the computation of this factor occurs when $\Delta\boldsymbol{\mu} = \mathbf{0}$, because under this condition $P(\Delta\mathbf{X} \geqslant \mathbf{0}) = P(\Delta\mathbf{Y} \geqslant \mathbf{0})$, where $\mathbf{Y} \sim N_n(\boldsymbol{\mu}, \Sigma)$, and thus:

$$P(\Delta\mathbf{X} \geqslant \mathbf{0}) = F_{n-1}(\mathbf{0}; \mathbf{0}, \Delta\Sigma\Delta^T, \varphi) = \Phi_{n-1}(\mathbf{0}; \mathbf{0}, \Delta\Sigma\Delta^T),$$

for any dispersion matrix Σ and characteristic generator φ. The condition $\Delta\boldsymbol{\mu} = \mathbf{0}$ holds, in particular, when $\mathbf{X} \sim EC_n(\boldsymbol{\mu}, \Sigma, \varphi)$ is an exchangeable elliptically contoured random vector, under which $\Delta\mathbf{X} \sim EC_{n-1}(\mathbf{0}, \sigma^2\Delta\Delta^T, \varphi)$, see Section 3, and so the exchangeability implies

$$P(\Delta\mathbf{X} \geqslant \mathbf{0}) = F_{n-1}(\mathbf{0}; \mathbf{0}, \Delta\Delta^T, \varphi) = \Phi_{n-1}(\mathbf{0}; \mathbf{0}, \Delta\Delta^T) = \frac{1}{n!},$$

for each $n \geqslant 2$ and elliptically contoured characteristic generator φ. Note that $\Delta\Delta^T = (\delta_{i,j})$, with $\delta_{i,i} = 2$, $\delta_{i-1,i} = \delta_{i+1,i} = -1$, and $\delta_{i,j} = 0$ otherwise.

3. L-statistics from exchangeable random variables

3.1. Exchangeable arbitrary distributions

A simple but important situation is when $\mathbf{X}$ is an exchangeable random vector, for instance such as in the genetic selection problem. This condition means that $P\mathbf{X} \stackrel{d}{=} \mathbf{X}$ for any permutation matrix $P \in \mathbb{R}^{n \times n}$, and thus each of the $n!$ different permutations $\mathbf{X}_i$ of $\mathbf{X}$ will have the same distribution as $\mathbf{X}$. Consider the selection defined by $\mathbf{X}^{(i)} = (\mathbf{X}_i | \Delta\mathbf{X}_i \geqslant \mathbf{0})$, $i = 1, \ldots, n!$, where $\Delta\mathbf{X} \geqslant \mathbf{0}$ means $X_2 - X_1 \geqslant 0, \ldots, X_n - X_{n-1} \geqslant 0$, and thus $X_1 \leqslant X_2 \leqslant \cdots \leqslant X_n$. Since $\Delta\mathbf{X}_i \geqslant \mathbf{0}$ is equivalent to $\mathbf{X}_i = \mathbf{X}_{(n)}$, it follows that $\mathbf{X}^{(i)} = \mathbf{X}_{(n)}$ if and only if $\mathbf{X}_i = \mathbf{X}_{(n)}$. Thus, when $\mathbf{X}$ is an exchangeable random vector, we have for all i that $\mathbf{X}_i \stackrel{d}{=} \mathbf{X}$ and so $\mathbf{X}^{(i)} \stackrel{d}{=} (\mathbf{X} | \Delta\mathbf{X} \geqslant \mathbf{0})$, implying that $\mathbf{X}_{(n)} \stackrel{d}{=} (\mathbf{X} | \Delta\mathbf{X} \geqslant \mathbf{0})$. In other words, under exchangeability of $\mathbf{X}$, the distribution of $\mathbf{X}_{(n)}$ is intimately related to the specific selection mechanism considered above. Next, we derive the pdf of $L\mathbf{X}_{(n)}$, for any fixed matrix $L \in \mathbb{R}^{p \times n}$, by assuming that the distribution of $\mathbf{X}$ is such that $L\mathbf{X}$ has a pdf denoted by $f_{L\mathbf{X}}$, only. Then, we focus on exchangeable elliptically contoured distributions, for which $L\mathbf{X}_{(n)} \stackrel{d}{=} (L\mathbf{X} | \Delta\mathbf{X} \geqslant \mathbf{0})$, implying that the distributions of L-statistics are in the *FUS* class (2).

When the random vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ is exchangeable, we have in (4) that $f_{L\mathbf{X}_i} = f_{L\mathbf{X}}$ and $P(\Delta\mathbf{X}_i \geqslant \mathbf{0}|L\mathbf{X}_i = \mathbf{y}) = P(\Delta\mathbf{X} \geqslant \mathbf{0}|L\mathbf{X} = \mathbf{y})$ for all i, yielding the following result based on Proposition 1.

Corollary 1. *Let $\mathbf{X} = (X_1, \ldots, X_n)^T$ be an absolutely continuous exchangeable random vector. Then, the pdf $f_{L\mathbf{X}_{(n)}}$ of $L\mathbf{X}_{(n)}$ is*

$$f_{L\mathbf{X}_{(n)}}(\mathbf{y}) = n! f_{L\mathbf{X}}(\mathbf{y}) P(\Delta\mathbf{X} \geqslant \mathbf{0}|L\mathbf{X} = \mathbf{y}), \quad \mathbf{y} \in \mathcal{S}_p, \tag{8}$$

where $f_{L\mathbf{X}}$ is the marginal pdf of $L\mathbf{X}$.

Because exchangeability implies $P(\Delta\mathbf{X} \geqslant \mathbf{0}) = 1/n!$, we can rewrite (8) as $f_{L\mathbf{X}_{(n)}}(\mathbf{y}) = f_{L\mathbf{X}}(\mathbf{y})$ $P(\Delta\mathbf{X} \geqslant \mathbf{0}|L\mathbf{X} = \mathbf{y})/P(\Delta\mathbf{X} \geqslant \mathbf{0})$ which is also the pdf of $(L\mathbf{X}|\Delta\mathbf{X} \geqslant \mathbf{0})$, and hence

$$L\mathbf{X}_{(n)} \stackrel{d}{=} (L\mathbf{X}|\Delta\mathbf{X} \geqslant \mathbf{0}) \tag{9}$$

in this setting. Corollary 1 is a generalization of a result given recently by Crocetta and Loperfido [9], who established a link to *FUSS* distributions in the case of i.i.d. components of $\mathbf{X}$. Their result is now a byproduct of Corollary 1.

3.2. Exchangeable elliptically contoured distributions

When $\mathbf{X} \sim EC_n(\boldsymbol{\mu}, \Sigma, \varphi)$ is an exchangeable elliptically contoured random vector, we have $\boldsymbol{\mu} = \boldsymbol{\mu}_n$ and $\Sigma = \Sigma_n$, where

$$\boldsymbol{\mu}_k = \mu\mathbf{1}_k, \quad \Sigma_k = \sigma^2(1-\rho)\Omega_k, \quad \Omega_k = I_k + \gamma\mathbf{1}_k\mathbf{1}_k^T, \ \ \gamma = \rho/(1-\rho), \ \ \rho \in [0, 1). \tag{10}$$

Rigorously speaking, the condition $\rho \in [0, 1)$ means that we are considering the exchangeability of an infinite sequence of elliptically contoured random variables. However, with the exception of the results in Section 3.3, most of the results given for such an infinite sequence can be applied to a finite sequence of exchangeable or permutation-symmetric elliptically contoured random variables, for which in (10) $\rho \in (-1/(n-1), 1)$ when $k = n$; see Tong [25, p. 109] for more details on this distinction. Since Σ_n has full rank n, we can assume that $\mathbf{X} \sim EC_n(\boldsymbol{\mu}_n, \Sigma_n, \varphi)$ has a pdf of the form (5), under which $L\mathbf{X} \sim EC_p(L\boldsymbol{\mu}_n, L\Sigma_n L^T, \varphi)$ has a pdf for any $L \in \mathbb{R}^{p\times n}$ of rank p, given by

$$f_{L\mathbf{X}}(\mathbf{y}) = f_p(\mathbf{y}; L\boldsymbol{\mu}_n, L\Sigma_n L^T, h^{(p)}) = \frac{|L\Omega_n L^T|^{-1/2} h^{(p)}(q(\mathbf{y}))}{\sigma^p(1-\rho)^{p/2}}, \quad \mathbf{y} \in \mathbb{R}^p, \tag{11}$$

where $q(\mathbf{y}) = \mathbf{u}^T(L\Omega_n L^T)^{-1}\mathbf{u}$, with $\mathbf{u} = (\mathbf{y} - L\boldsymbol{\mu}_n)/(\sigma\sqrt{1-\rho})$. Moreover, since $\Delta\mathbf{1}_n = \mathbf{0}$, and so $\Delta\Sigma_n = \sigma^2(1-\rho)\Delta$, we have from the properties of elliptically contoured distributions that $\Delta\mathbf{X}|L\mathbf{X} = \mathbf{y}$ has a distribution given by

$$EC_n(\Delta L^T(L\Omega_n L^T)^{-1}(\mathbf{y} - L\boldsymbol{\mu}_n), \sigma^2(1-\rho)\Delta\{I_n - L^T(L\Omega_n L^T)^{-1}L\}\Delta^T, \varphi_{q(\mathbf{y})}),$$

implying that

$$P(\Delta\mathbf{X} \geqslant \mathbf{0}|L\mathbf{X} = \mathbf{y}) = F_{n-1}(\Delta L^T(L\Omega_n L^T)^{-1}\mathbf{u}; \mathbf{0}, \Delta\{I_n - L^T(L\Omega_n L^T)^{-1}L\}\Delta^T, \varphi_{q(\mathbf{y})}).$$

Therefore, we have the following result based on Proposition 2.

Corollary 2. *Let* $\mathbf{X} = (X_1, \ldots, X_n)^T \sim EC_n(\boldsymbol{\mu}_n, \Sigma_n, \varphi)$ *be an exchangeable elliptically contoured random vector, i.e. with* $\boldsymbol{\mu}_n$ *and* Σ_n *as in* (10), *and assume it has a pdf of the form* (5). *Then, for any* $L \in \mathbb{R}^{p \times n}$ *of rank* p $(1 \leqslant p \leqslant n)$, *the pdf* $f_{L\mathbf{X}_{(n)}}$ *of* $L\mathbf{X}_{(n)}$ *is*

$$f_{L\mathbf{X}_{(n)}}(\mathbf{y}) = n!\, f_p(\mathbf{y}; L\boldsymbol{\mu}_n, L\Sigma_n L^T, h^{(p)}) F_{n-1}(\Delta L^T (L\Omega_n L^T)^{-1}\mathbf{u};$$
$$\mathbf{0}, \Delta\{I_n - L^T(L\Omega_n L^T)^{-1}L\}\Delta^T, \varphi_{q(\mathbf{y})}), \quad \mathbf{y} \in \mathcal{S}_p, \tag{12}$$

where $q(\mathbf{y}) = \mathbf{u}^T(L\Omega_n L^T)^{-1}\mathbf{u}$, *with* $\mathbf{u} = (\mathbf{y} - L\boldsymbol{\mu}_n)/(\sigma\sqrt{1-\rho})$ *and* Ω_n *as in* (10).

When $\rho = 0$, (12) corresponds to the pdf of $L\mathbf{X}_{(n)}$ under a spherically contoured distribution $EC_n(\mu\mathbf{1}_n, \sigma^2 I_n, \varphi)$ for $\mathbf{X}$. Moreover, if φ is the normal characteristic generator, then we obtain the pdf corresponding to the L-statistics of i.i.d. $N(\mu, \sigma^2)$ random variables.

When $L\mathbf{1}_n = \mathbf{0}$, we have that $L\Sigma_n L^T = \sigma^2(1-\rho)LL^T$, and the pdf of $L\mathbf{X}_{(n)}$ can be obtained assuming $\mathbf{X} \sim EC_n(\mathbf{0}, \sigma^2(1-\rho)I_n, \varphi)$. Since there are many important L-statistics satisfying the condition $L\mathbf{1}_n = \mathbf{0}$, e.g. such as the range $L\mathbf{X}_{(n)} = X_{(n)} - X_{(1)}$, we establish the following result (see also Corollary 4 in the next subsection).

Corollary 3. *Let* $L\mathbf{X}_{(n)}$ *be the L-statistics induced by an exchangeable elliptically contoured random vector* $\mathbf{X} \sim EC_n(\mu\mathbf{1}_n, \sigma^2\{(1-\rho)I_n + \rho\mathbf{1}_n\mathbf{1}_n^T\}, \varphi)$, *satisfying the condition* $L\mathbf{1}_n = \mathbf{0}$. *Then* $L\mathbf{X}_{(n)} \overset{d}{=} \sigma\sqrt{1-\rho}L\mathbf{U}_{(n)}$, *where* $\mathbf{U}_{(n)}$ *is the vector of order statistics induced by the spherically contoured random vector* $\mathbf{U} \sim EC_n(\mathbf{0}, I_n, \varphi)$, $\rho \in [0, 1)$.

Finally, when $p = 1$ we have univariate L-statistics of the form $L\mathbf{X}_{(n)} = \sum_{i=1}^n a_i X_{(i)}$, which by (12) have a pdf of the form

$$f_{L\mathbf{X}_{(n)}}(y) = n!\, f_1(y; \eta, \tau^2, h^{(1)}) F_{n-1}(z\Delta\mathbf{a}_\gamma; \mathbf{0}, \Delta\{I_n - \mathbf{a}_\gamma\mathbf{a}_\gamma^T\}\Delta^T, h_{z^2}^{(n-1)}), \quad y \in \mathbb{R}, \tag{13}$$

where $\eta = \mu\sum_{i=1}^n a_i$, $\tau^2 = \sigma^2(1-\rho)\{\sum_{i=1}^n a_i^2 + \gamma(\sum_{i=1}^n a_i)^2\}$, $z = (y-\eta)/\tau$ and $\mathbf{a}_\gamma = \mathbf{a}/\{\sum_{i=1}^n a_i^2 + \gamma(\sum_{i=1}^n a_i)^2\}^{1/2}$, with $\gamma = \rho/(1-\rho)$. Thus, from (13) we can obtain, e.g. the pdf of the sum of the two best $X_{(n-1)} + X_{(n)}$, the range $X_{(n)} - X_{(1)}$, the midpoint $(X_{(1)} + X_{(n)})/2$, the median when n is even $(X_{(n/2)} + X_{(n/2+1)})/2$, and so on. For example, for the sum of the two best $X_{(n-1)} + X_{(n)}$, we have $\mathbf{a}_\gamma = (\mathbf{e}_{n-1} + \mathbf{e}_n)/\sqrt{2+4\gamma}$, $\eta = 2\mu$ and $\tau^2 = 2(1+\rho)\sigma^2$, and by (13) its pdf is

$$f_{X_{(n-1)}+X_{(n)}}(y) = n!\, f_1(y; 2\mu, 2(1+\rho)\sigma^2, h^{(1)}) F_{n-1}(z\Delta(\mathbf{e}_{n-1} + \mathbf{e}_n);$$
$$\mathbf{0}, \Delta\{I_n - [1/(2+4\gamma)](\mathbf{e}_{n-1}+\mathbf{e}_n)(\mathbf{e}_{n-1}+\mathbf{e}_n)^T\}\Delta^T, h_{z^2}^{(n-1)}), \quad y \in \mathbb{R}, \tag{14}$$

where $z = (y - 2\mu)\sqrt{(1-\rho)/(1+\rho)}/(2\sigma)$. Moreover, as indicated in Corollary 3, those cases with $L\mathbf{1}_n = \sum_{i=1}^n a_i = 0$ yield additional simplifications, since $\eta = 0$ and $\mathbf{a}_\gamma = \mathbf{a}_0 = \mathbf{a}/\sqrt{\sum_{i=1}^n a_i^2}$. For instance, the range $L\mathbf{X}_{(n)} = X_{(n)} - X_{(1)}$ yields $\eta = 0$, $\tau^2 = 2\sigma^2(1-\rho)$ and $\mathbf{a}_0 = (\mathbf{e}_n - \mathbf{e}_1)/\sqrt{2}$. By (13), the pdf of the range is

$$f_{X_{(n)}-X_{(1)}}(y) = n!\, f_1(y; 0, 2(1-\rho)\sigma^2, h^{(1)}) F_{n-1}(z\Delta(\mathbf{e}_n - \mathbf{e}_1);$$
$$\mathbf{0}, \Delta\{I_n - (1/2)(\mathbf{e}_n - \mathbf{e}_1)(\mathbf{e}_n - \mathbf{e}_1)^T\}\Delta^T, h_{z^2}^{(n-1)}), \quad y \in \mathbb{R}, \tag{15}$$

where $z = y/(2\sigma\sqrt{1-\rho})$.

Remark 1. If the n-dimensional unit basis vectors $\mathbf{e}_i$'s are denoted by $\mathbf{e}_{n,i}$'s to indicate explicitly that they have dimension n, then in the computation of $\Delta\mathbf{e}_{n,i}$, we can use the fact that $\Delta\mathbf{e}_{n,1} = -\mathbf{e}_{n-1,1}$, $\Delta\mathbf{e}_{n,n} = \mathbf{e}_{n-1,n-1}$ and $\Delta\mathbf{e}_{n,i} = \mathbf{e}_{n-1,i-1} - \mathbf{e}_{n-1,i}$, for $i = 2, \ldots, n-1$.

3.3. Stochastic representation

Let $U_0, U_1, \ldots, U_n$ be uncorrelated standardized random variables and let

$$\mathbf{X} = \sigma\{\sqrt{1-\rho}\,\mathbf{U} + \sqrt{\rho}\,\mathbf{1}_n U_0\} + \mu\mathbf{1}_n, \quad \rho \in [0, 1), \tag{16}$$

where $\mathbf{U} = (U_1, \ldots, U_n)^T$. Let $P \in \mathbb{R}^{n\times n}$ be a permutation matrix, i.e. $P\mathbf{1}_n = \mathbf{1}_n$. It follows from (16) that $P\mathbf{X} \stackrel{d}{=} \mathbf{X} \Leftrightarrow P\mathbf{U} \stackrel{d}{=} \mathbf{U}$; in other words, $\mathbf{X}$ is exchangeable if and only if $\mathbf{U}$ is exchangeable. Since $PP^T = I_n$, this fact holds, e.g., when $\mathbf{U}$ is a spherically contoured random vector. Thus, (16) holds also under the following situations:

(i) $U_0, U_1, \ldots, U_n$ are i.i.d. random variables;
(ii) $\mathbf{U} \sim EC_n(\mathbf{0}, I_n, \varphi)$ and is independent of U_0;
(iii) $(U_0, \mathbf{U}^T)^T \sim EC_{n+1}(\mathbf{0}, I_{n+1}, \varphi)$.

From (16) it is immediate that

$$L\mathbf{X}_{(n)} \stackrel{d}{=} \sigma\{\sqrt{1-\rho}\,L\mathbf{U}_{(n)} + \sqrt{\rho}\,L\mathbf{1}_n U_0\} + \mu L\mathbf{1}_n, \quad \rho \in [0, 1), \tag{17}$$

where $\mathbf{U}_{(n)} = (U_{(1)}, \ldots, U_{(n)})^T$ is the vector of order statistics induced by $\mathbf{U} = (U_1, \ldots, U_n)^T$. Thus, if we consider situation (iii), then (16) implies $\mathbf{X} \sim EC_n(\mu\mathbf{1}_n, \sigma^2\{(1-\rho)I_n + \rho\mathbf{1}_n\mathbf{1}_n^T\}, \varphi)$, and $L\mathbf{X}_{(n)}$ has pdf given by (12). For the other situations specified in (i) and (ii), we can apply the more general result (8) to derive the conditional pdf of $L\mathbf{U}_{(n)}$ given $U_0 = x_0$, and therefore obtain the pdf of $L\mathbf{X}_{(n)}$ as follows.

Proposition 3. *Let $f_{L\mathbf{U}|U_0=x_0}$ be the conditional pdf of $L\mathbf{U}$ given $U_0 = x_0$ and f_{U_0} be the pdf of U_0. The pdf of $L\mathbf{X}_{(n)}$ is*

$$f_{L\mathbf{X}_{(n)}}(\mathbf{y}) = \frac{n!}{\sigma^p(1-\rho)^{p/2}} \int_{-\infty}^{\infty} f_{L\mathbf{U}|U_0=x_0}(\mathbf{u} - \sqrt{\gamma}\,L\mathbf{1}_n x_0) \times P(\Delta\mathbf{U} \geqslant \mathbf{0} | L\mathbf{U} = \mathbf{u} - \sqrt{\gamma}\,L\mathbf{1}_n x_0, U_0 = x_0) f_{U_0}(x_0)\,dx_0, \quad \mathbf{y} \in \mathcal{S}_p, \tag{18}$$

where $\mathbf{u} = (\mathbf{y} - \mu L\mathbf{1}_n)/(\sigma\sqrt{1-\rho})$ and $\gamma = \rho/(1-\rho)$.

Proof. Conditioning (17) on $U_0 = x_0$, we have by (9) that

$$\begin{aligned}(L\mathbf{X}_{(n)}|U_0 = x_0) &\stackrel{d}{=} (L\mathbf{X}|\Delta\mathbf{X} \geqslant \mathbf{0}, U_0 = x_0) \\ &= \sigma\sqrt{1-\rho}\,(L\mathbf{U}|\Delta\mathbf{U} \geqslant \mathbf{0}, U_0 = x_0) + \sigma\sqrt{\rho}\,x_0 L\mathbf{1}_n + \mu L\mathbf{1}_n \\ &\stackrel{d}{=} \sigma\sqrt{1-\rho}\,(L\mathbf{U}_{(n)}|U_0 = x_0) + \sigma\sqrt{\rho}\,x_0 L\mathbf{1}_n + \mu L\mathbf{1}_n,\end{aligned}$$

which implies that

$$f_{L\mathbf{X}_{(n)}}(\mathbf{y}) = \frac{1}{\sigma^p(1-\rho)^{p/2}} \int_{-\infty}^{\infty} f_{L\mathbf{U}_{(n)}|U_0=x_0}[(\mathbf{y} - \sigma\sqrt{\rho}\, x_0 L\mathbf{1}_n - \mu L\mathbf{1}_n)/(\sigma\sqrt{1-\rho})] \times f_{U_0}(x_0)\, dx_0,$$

$\mathbf{y} \in \mathcal{S}_p$. Thus, the result follows by applying (8) to obtain $f_{L\mathbf{U}_{(n)}|U_0=x_0}$ and then replace it in the last expression above, where we note also that

$$\begin{aligned} &P(\Delta\mathbf{X} \geqslant \mathbf{0} | L\mathbf{X} = \mathbf{y}, U_0 = x_0) \\ &\quad = P(\Delta\mathbf{U} \geqslant \mathbf{0} | L\mathbf{U} = (\mathbf{y} - \sigma\sqrt{\rho}\, x_0 L\mathbf{1}_n - \mu L\mathbf{1}_n)/(\sigma\sqrt{1-\rho}), U_0 = x_0). \qquad \square \end{aligned}$$

This approach yields significant simplifications in the derivation of the distribution of L-statistics satisfying $L\mathbf{1}_n = \mathbf{0}$. In fact, by (17) we have that $L\mathbf{X}_{(n)} \overset{d}{=} \sigma\sqrt{1-\rho} L\mathbf{U}_{(n)}$, so that (18) reduces to

$$\begin{aligned} f_{L\mathbf{X}_{(n)}}(\mathbf{y}) &= \frac{1}{\sigma^p(1-\rho)^{p/2}} f_{L\mathbf{U}_{(n)}}(\mathbf{y}/\sigma\sqrt{1-\rho}) \\ &= \frac{n!}{\sigma^p(1-\rho)^{p/2}} f_{L\mathbf{U}}(\mathbf{y}/\sigma\sqrt{1-\rho}) P(\Delta\mathbf{U} \geqslant \mathbf{0} | L\mathbf{U} = \mathbf{y}/(\sigma\sqrt{1-\rho})), \\ &\quad \mathbf{y} \in \mathcal{S}_p. \end{aligned} \tag{19}$$

Note that this fact does not depend on the distribution of U_0 and holds whatever the distribution of the exchangeable random vector $\mathbf{U}$. Therefore, we obtain the following extension of the result given in Corollary 3 for the specific case of the elliptically contoured family.

Corollary 4. *Let $L\mathbf{X}_{(n)}$ be the L-statistics defined by* (17). *If $L\mathbf{1}_n = \mathbf{0}$, then $L\mathbf{X}_{(n)}$ and $\sigma\sqrt{1-\rho} L\mathbf{U}_{(n)}$ have the same distribution, with pdf given by* (19).

This approach yields the results obtained by Tong [25] for the normal distribution of $\mathbf{X}$. Tong [25] extended also these results to the Student t distribution by using its representation as a scale mixture of the normal distribution.

Another important application of (17) is that it can be used to obtain the moments of $L\mathbf{X}_{(n)}$, following Afonja [1] who considered the special case of the normal and Student t distributions.

4. Joint distribution of p order statistics

4.1. Dependent random variables

We consider now one of the most common L-statistics studied in the literature under the particular case of i.i.d. random variables and also under some specific exchangeable parent distributions such as the normal one (see, e.g. [25]). Specifically, for any matrix $L \in \mathbb{R}^{p\times n}$ such that $L\mathbf{1}_n = \mathbf{1}_p$ and $LL^T = I_p$, i.e. $L = (\mathbf{e}_{r_1}, \ldots, \mathbf{e}_{r_p})$, where $\mathbf{e}_j$ denotes the jth n-dimensional unit basis vector, we consider L-statistics of the form

$$L\mathbf{X}_{(n)} = (X_{(r_1)}, \ldots, X_{(r_p)})^T, \quad 1 \leqslant r_1 < \cdots < r_p \leqslant n,$$

for which a substantial reduction in the computation of their pdf (4) is possible whatever the absolutely continuous distribution of $\mathbf{X}=(X_1,\ldots,X_n)^T$. In fact, note first that $L\mathbf{X}=(X_{i_1},\ldots,X_{i_p})^T$ and $\mathcal{S}_p=\{\mathbf{y}\in\mathbb{R}^p : -\infty<y_1\leqslant\cdots\leqslant y_p<\infty\}$ by (3). Now, there are $n_p=n!/(n-p)!$ ways of choosing the p components, say $(X_{i_1},\ldots,X_{i_p})$, from $(X_1,\ldots,X_n)$, and there are

$$k_p=\binom{n-p}{r_1-1}\left[\prod_{j=2}^{p}\binom{n-p-r_j+1}{r_j-r_{j-1}-1}\right]$$

permutation vectors $\mathbf{X}_j$ of $\mathbf{X}$ for which $(X_{(r_1)},\ldots,X_{(r_p)})=(X_{i_1},\ldots,X_{i_p})$, i.e. such that $\Delta\mathbf{X}_j\geqslant\mathbf{0}$ given $L\mathbf{X}_i=(X_{i_1},\ldots,X_{i_p})^T$. Hence, for any $1\leqslant r_1<\cdots<r_p\leqslant n$ and $\mathbf{y}=(y_1,\ldots,y_p)^T\in\mathcal{S}_p$, we have

$$f_{L\mathbf{X}_{(n)}}(\mathbf{y})=\sum_{i=1}^{n_p}f_{L\mathbf{X}_i}(\mathbf{y})\sum_{j=1}^{k_p}P(\Delta\mathbf{X}_j\geqslant\mathbf{0}|L\mathbf{X}_i=\mathbf{y}),$$

that is,

$$\begin{aligned}f_{X_{(r_1)},\ldots,X_{(r_p)}}(y_1,\ldots,y_p)=&\sum_{i_1\neq\cdots\neq i_p}f_{X_{i_1},\ldots,X_{i_p}}(y_1,\ldots,y_p)\\&\times\sum_{(i_1,\ldots,i_p)}P(X_{j_1}\leqslant X_{i_1},\ldots,X_{j_{r_1-1}}\leqslant X_{i_1},\\&X_{i_1}<X_{j_{r_1+1}}\leqslant X_{i_2},\ldots,X_{i_1}<X_{j_{r_2-1}}\leqslant X_{i_2},\ldots,\\&X_{i_{p-1}}<X_{j_{r_{p-1}+1}}\leqslant X_{i_p},\ldots,X_{i_{p-1}}<X_{j_{r_p-1}}\leqslant X_{i_p},\\&X_{j_{r_p+1}}>X_{i_p},\ldots,X_{j_n}>X_{i_p}|X_{i_1}=y_1,\ldots,X_{i_p}=y_p),\end{aligned}\tag{20}$$

where $\sum_{(i_1,\ldots,i_p)}$ denotes the summation over all k_p sets of indices $(j_1\ldots,j_{r_1-1},j_{r_2+1},\ldots,j_{r_p-1},j_{r_p+1},\ldots,j_n)$ such that $1\leqslant j_1<\cdots<j_{r_1-1}\leqslant n$, $1\leqslant j_{r_1+1}<\cdots<j_{r_2-1}\leqslant n,\ldots,1\leqslant j_{r_p+1}<\cdots<j_n\leqslant n$, and $j_{m_1}\neq\cdots\neq j_{m_{p+1}}\neq i_m$ $m_1=1,\ldots,r_1-1,\ldots,m_{p+1}=r_p+1,\ldots,n$, $m=i_1,\ldots,i_p$.

When p=1, we have for any $1\leqslant r\leqslant n$ and $-\infty<x<\infty$ that

$$f_{X_{(r)}}(x)=\sum_{i=1}^{n}f_{X_i}(x)\sum_{(i)}P(X_{j_1}\leqslant X_i,\ldots,X_{j_{r-1}}\leqslant X_i,X_{j_{r+1}}>X_i,\ldots,X_{j_n}>X_i|X_i=x),$$

where $\sum_{(i)}$ denotes the summation over all

$$\binom{n-1}{r-1}$$

sets of indices $(j_1,\ldots,j_{r-1},j_{r+1},\ldots,j_n)$ such that $1\leqslant j_1<\cdots<j_{r-1}\leqslant n$,$1\leqslant j_{r+1}<\cdots<j_n\leqslant n$, and $j_k\neq j_m\neq i,k=1,\ldots,r-1,m=r+1,\ldots,n$. The particular case with $r=n$ yields the distribution of $X_{(n)}=\max\{X_1,\ldots,X_n\}$, which was studied recently by Arellano-Valle and Genton [4].

When $p=2$ we have for any $1 \leqslant r < s \leqslant n$ and $-\infty < x \leqslant y < \infty$ that

$$f_{X_{(r)},X_{(s)}}(x,y) = \sum_{i\neq j} f_{X_i,X_j}(x,y) \sum\nolimits_{(i,j)} P(X_{j_1} \leqslant X_i, \ldots, X_{j_{r-1}} \leqslant X_i,$$
$$X_i < X_{j_{r+1}} \leqslant X_j, \ldots, X_i < X_{j_{s-1}} \leqslant X_j,$$
$$X_{j_{s+1}} > X_j, \ldots, X_{j_n} > X_j | X_i = x, X_j = y),$$

where $\sum_{(i,j)}$ denotes the summation over all

$$\binom{n-2}{r-1}\binom{n-r-1}{s-r-1}$$

sets of indices $(j_1, \ldots, j_{r-1}, j_{r+1}, \ldots, j_{s-1}, j_{s+1}, \ldots, j_n)$ such that $1 \leqslant j_1 < \cdots < j_{r-1} \leqslant n$, $1 \leqslant j_{r+1} < \cdots < j_{s-1} \leqslant n$, $1 \leqslant j_{s+1} < \cdots < j_n \leqslant n$, and $j_k \neq j_l \neq j_m \neq i, j$, $k = 1, \ldots, r-1$, $l = r+1, \ldots, s-1$, $m = s+1, \ldots, n$.

4.2. Exchangeable random variables

When $\mathbf{X} = (X_1, \ldots, X_n)^T$ is exchangeable, (20) allows us to express the pdf (8) for $L\mathbf{X}_{(n)} = (X_{(r_1)}, \ldots, X_{(r_p)})^T$, $1 \leqslant r_1 < \cdots < r_p \leqslant n$, in terms of a conditional probability on $\mathbb{R}^{n-p}$, as is indicated next. Let

$$\mathbf{Y}_0 = L\mathbf{X} = (X_{r_1}, \ldots, X_{r_p})^T \quad \text{and} \quad \mathbf{Y}_s = A_s\mathbf{X} = (X_{r_{s-1}+1}, \ldots, Y_{r_s-1})^T, \quad s = 1, \ldots, p+1,$$

where $r_0 = 0$, $r_{p+1} = n+1$ and the A_s's are matrices of dimensions $(r_s - r_{s-1} - 1) \times n$ such that

$$A_s\mathbf{1}_n = \mathbf{1}_{r_s - r_{s-1}-1}, \quad A_sA_s^T = I_{r_s-r_{s-1}-1}, \quad LA_s^T = \mathbf{0} \quad \text{and} \quad A_sA_{s'}^T = \mathbf{0}, \quad s \neq s'. \tag{21}$$

Note for any $\mathbf{y} = (y_1, \ldots, y_p)^T$, with $-\infty < y_1 \leqslant \cdots \leqslant y_p < \infty$, that

$$P(\Delta\mathbf{X} \geqslant \mathbf{0} | L\mathbf{X} = \mathbf{y}) = P(X_1 \leqslant \cdots \leqslant X_n | X_{r_1} = y_1, \ldots, X_{r_p} = y_p)$$
$$= \frac{Q_{n-p}(\mathbf{y})}{(r_1-1)![\prod_{j=2}^{p}(r_j - r_{j-1} - 1)!](n-r_p)!},$$

where

$$Q_{n-p}(\mathbf{y}) = P(\mathbf{Y}_1 \leqslant \mathbf{1}_{r_1-1}y_1, \mathbf{1}_{r_2-r_1-1}y_1 < \mathbf{Y}_2 \leqslant \mathbf{1}_{r_2-r_1-1}y_2, \ldots,$$
$$\mathbf{1}_{r_p-i_{p-1}-1}y_{p-1} < \mathbf{Y}_{p-1} \leqslant \mathbf{1}_{r_p-r_{p-1}-1}y_p, \mathbf{1}_{n-r_p}y_p < \mathbf{Y}_{p+1} | \mathbf{Y}_0 = \mathbf{y}), \tag{22}$$

yielding the following result.

Corollary 5. *Let $\mathbf{X} = (X_1, \ldots, X_n)^T$ be an exchangeable random vector, and consider a matrix $L \in \mathbb{R}^{p\times n}$ such that $L\mathbf{1}_n = \mathbf{1}_p$, $LL^T = I_p$ and $L\mathbf{X} = (X_{r_1}, \ldots, X_{r_p})^T$, with $1 \leqslant r_1 < \cdots < r_p \leqslant n$. Then, for any $\mathbf{y} = (y_1, \ldots, y_p)^T$, with $-\infty < y_1 \leqslant \cdots \leqslant y_p < \infty$, the pdf of $L\mathbf{X}_{(n)} = (X_{(r_1)}, \ldots, X_{(r_p)})^T$ is*

$$f_{L\mathbf{X}_{(n)}}(\mathbf{y}) = c_n f_{L\mathbf{X}}(\mathbf{y}) Q_{n-p}(\mathbf{y}), \tag{23}$$

where $Q_{n-p}(\mathbf{y})$ *is given by* (22) *and*

$$c_n = \frac{n!}{(r_1-1)![\prod_{k=2}^{p}(r_k-r_{k-1}-1)!](n-r_p)!}.$$

When $\mathbf{Y}_0, \mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_{p+1}$ are independent random vectors, we have

$$Q_{n-p}(\mathbf{y}) = P(\mathbf{Y}_1 \leqslant \mathbf{1}_{r_1-1}y_{r_1})P(\mathbf{1}_{r_2-r_1-1}y_1 < \mathbf{Y}_2 \leqslant \mathbf{1}_{r_2-r_1-1}y_{r_2})$$
$$\cdots P(\mathbf{1}_{r_p-r_{p-1}-1}y_{r_{p-1}} < \mathbf{Y}_{r_{p-1}} \leqslant \mathbf{1}_{r_p-r_{p-1}-1}y_{r_p})P(\mathbf{1}_{n-r_p}y_{r_p} < \mathbf{Y}_{p+1}).$$

Hence, when the X_i's are i.i.d. random variables with cdf F, this probability reduces to

$$[F(y_1)]^{r_1-1}\left(\prod_{k=2}^{p}\left[F(y_k)-F(y_{k-1})\right]^{r_k-r_{k-1}-1}\right)[1-F(y_p)]^{n-r_p}.$$

From (23), we obtain the following well-known joint pdf for $X_{(r_1)} \leqslant \cdots \leqslant X_{(r_p)}$ when F has pdf f:

$$f_{X_{(r_1)},\ldots,X_{(r_p)}}(y_1,\ldots,y_p) = c_n\left(\prod_{j=1}^{p} f(y_j)\right)[F(y_1)]^{r_1-1}$$
$$\times\left(\prod_{k=2}^{p}\left[F(y_k)-F(y_{k-1})\right]^{r_k-r_{k-1}-1}\right)[1-F(y_p)]^{n-r_p},$$

where $-\infty < y_1 \leqslant \cdots \leqslant y_p < \infty$.

When $p=1$, we have $Q_{n-1}(y)=P(\mathbf{Y}_1 \leqslant y\mathbf{1}_{r-1}, \mathbf{Y}_2 > y\mathbf{1}_{n-r}|X_r=y)$, with $\mathbf{Y}_1=(X_1,\ldots,X_{r-1})^T$ and $\mathbf{Y}_2=(X_{r+1},\ldots,X_n)^T$, so that (23) reduces to

$$f_{X_{(r)}}(y) = \frac{n!}{r_1!(n-r)!}f_{X_r}(y)P(\mathbf{Y}_1 \leqslant y\mathbf{1}_{r-1}, \mathbf{Y}_2 > y\mathbf{1}_{n-r}|X_r=y), \quad y \in \mathbb{R}.$$

When $\mathbf{X}=(X_1,\ldots,X_n)^T$ has an exchangeable elliptically contoured distribution, we have

$$\mathbf{Y}_0 = L\mathbf{X} = (X_{r_1},\ldots,X_{r_p})^T \sim EC_p(\boldsymbol{\mu}_p, \Sigma_p, \varphi),$$

where by (10) $\boldsymbol{\mu}_p = \mu\mathbf{1}_p$ and $\Sigma_p = \sigma^2(1-\rho)\Omega_p$, with $\Omega_p = I_p + \gamma\mathbf{1}_p\mathbf{1}_p^T$, since $L\mathbf{1}_n = \mathbf{1}_p$ and $LL^T = I_p$, and so $L\boldsymbol{\mu}_n = \boldsymbol{\mu}_p$ and $L\Omega_nL^T = \Omega_p$. Thus, if we assume that $\mathbf{X}$ has a pdf, then by (11)

$$f_{L\mathbf{X}}(\mathbf{y}) = f_p(\mathbf{y};\boldsymbol{\mu}_p,\Sigma_p,h^{(p)}) = \frac{|\Omega_p|^{-1/2}h^{(p)}(q_p(\mathbf{y}))}{\sigma^p(1-\rho)^{p/2}}, \quad \mathbf{y} \in \mathbb{R}^p,$$

where $q_p(\mathbf{y})=\mathbf{u}^T\Omega_p^{-1}\mathbf{u}$, with $\mathbf{u}=(\mathbf{y}-\boldsymbol{\mu}_p)/(\sigma\sqrt{1-\rho})$. Moreover, writing $\mathbf{Y}=(\mathbf{Y}_1^T,\ldots,\mathbf{Y}_{p+1}^T)^T = A\mathbf{X}$, with $A=\mathrm{diag}\{A_1,\ldots,A_{p+1}\}$, we have by (21) and some standard computations that

$$(\mathbf{Y}|L\mathbf{X}=\mathbf{y}) \sim EC_{n-p}(\eta_p\mathbf{1}_{n-p}, \sigma^2(1-\rho)\Gamma_{n-p}, \varphi_{q_p(\mathbf{y})}),$$

where $\eta_p = (1-p\gamma_p)\mu + p\gamma_p\bar{y}_p$ and $\Gamma_{n-p} = I_{n-p} + \gamma_p\mathbf{1}_{n-p}\mathbf{1}_{n-p}^T$, with $\bar{y}_p = (\sum_{i=1}^{p} y_i)/p$ and $\gamma_p = \rho/[1+(p-1)\rho]$. Thus, letting $\mathbf{U} = (\mathbf{Y}-\eta_p\mathbf{1}_{n-p})/(\sigma\sqrt{1-\rho})$ in (22), we have

$$Q_{n-p}(\mathbf{y}) = P(\mathbf{U} \in \mathcal{C}|L\mathbf{X}=\mathbf{y}) = \int_{\mathcal{C}} f_{n-p}(\mathbf{v};\mathbf{0},\Gamma_{n-p},h_{q_p(\mathbf{y})}^{(n-p)})\,d\mathbf{v},$$

since $(\mathbf{U}|\mathbf{Y}_0=\mathbf{y}) \sim EC_{n-p}(\mathbf{0}, \Gamma_{n-p}, \varphi_{q_p(\mathbf{y})})$. Here for any partition $\mathbf{v}=(\mathbf{v}_1^T, \ldots, \mathbf{v}_{p+1}^T)^T$, with $\mathbf{v}_s \in \mathbb{R}^{r_s-r_{s-1}-1}$, $s=1, \ldots, p+1$, $\mathcal{C}$ is the subset of $\mathbb{R}^{n-p}$ given by

$$\mathcal{C}=\{\mathbf{v}_1 \leqslant \mathbf{1}_{r_1-1}u_1, \mathbf{1}_{r_2-r_1-1}u_1<\mathbf{v}_2 \leqslant \mathbf{1}_{r_2-r_1-1}u_2, \ldots,$$
$$\mathbf{1}_{r_p-i_{p-1}-1}u_{p-1}<\mathbf{v}_{p-1} \leqslant \mathbf{1}_{r_p-r_{p-1}-1}u_p, \mathbf{1}_{n-r_p}u_p<\mathbf{v}_{p+1}\},$$

with $u_i=(y_i-\eta_p)/(\sigma\sqrt{1-\rho})$, $i=1, \ldots, p$. In other words, when $\mathbf{X}$ has an exchangeable elliptically contoured distribution, the pdf (23) of $L\mathbf{X}_{(n)}=(X_{(r_1)}, \ldots, X_{(r_p)})^T$ can be expressed as

$$f_{L\mathbf{X}_{(n)}}(\mathbf{y})=c_n f_p(\mathbf{y}; \boldsymbol{\mu}_p, \Sigma_p, h^{(p)}) \int_{\mathcal{C}} f_{n-p}(\mathbf{v}; \mathbf{0}, \Gamma_{n-p}, h^{(n-p)}_{q_p(\mathbf{y})})\, d\mathbf{v},$$

for any $\mathbf{y}=(y_1, \ldots, y_p)^T$, with $-\infty<y_1 \leqslant \cdots \leqslant y_p<\infty$. For $p=1$, this yields the following pdf for $X_{(r)}$, the rth order statistic, $r=1, \ldots, n$:

$$f_{X_{(r)}}(y)=\frac{n!}{(r-1)!(n-r)!} f_1(y; \mu, \sigma^2, h^{(1)})$$
$$\times F_{n-1}(z\sqrt{1-\rho}\mathbf{J}_{n-1}; \mathbf{0}, I_{n-1}+\rho\mathbf{1}_{n-1}\mathbf{1}_{n-1}^T, h^{(n-1)}_{z^2}), \quad y \in \mathbb{R},$$

where $z=(y-\mu)/\sigma$ and $\mathbf{J}_{n-1}=(\mathbf{1}_{r-1}^T, -\mathbf{1}_{n-r}^T)^T$. For example, if $h^{(n)}(u)=(2\pi)^{-n/2}\exp\{-u/2\}$, i.e. the n-dimensional normal generator, then

$$f_{L\mathbf{X}_{(n)}}(\mathbf{y})=c_n\phi_p(\mathbf{y}; \boldsymbol{\mu}_p, \Sigma_p) \int_{\mathcal{C}} \phi_{n-p}(\mathbf{u}; \mathbf{0}, \Gamma_{n-p})\, d\mathbf{u},$$

and for $p=1$,

$$f_{X_{(r)}}(y)=\frac{n!}{(r-1)!(n-r)!}\phi_1(y; \mu, \sigma^2)\Phi_{n-1}(z\sqrt{1-\rho}\mathbf{J}_{n-1}; \mathbf{0}, I_{n-1}+\rho\mathbf{1}_{n-1}\mathbf{1}_{n-1}^T),$$
$$y \in \mathbb{R}.$$

Note, however, that for $p=1$, (12) yields the following pdf for $X_{(r)}$, the rth order statistic, $r=1, \ldots, n$:

$$f_{X_{(r)}}(y)=n! f_1(y; \mu, \sigma^2, h^{(1)})F_{n-1}(z\sqrt{1-\rho}\Delta\mathbf{e}_r; \mathbf{0}, \Delta\{I_n-(1-\rho)\mathbf{e}_r\mathbf{e}_r^T\}\Delta^T, h^{(n-1)}_{z^2}),$$
$$y \in \mathbb{R},$$

where $z=(y-\mu)/\sigma$ and $\mathbf{e}_r$ is the rth n-dimensional unit basis vector, implying for $r=1, \ldots, n$:

$$F_{n-1}(z\sqrt{1-\rho}\Delta\mathbf{e}_i; \mathbf{0}, \Delta\{I_n-(1-\rho)\mathbf{e}_i\mathbf{e}_i^T\}\Delta^T, h^{(n-1)}_{z^2})$$
$$=\frac{F_{n-1}(z\sqrt{1-\rho}\mathbf{J}_{n-1}; \mathbf{0}, I_{n-1}+\rho\mathbf{1}_{n-1}\mathbf{1}_{n-1}^T, h^{(n-1)}_{z^2})}{(r-1)!(n-r)!}, \quad z \in \mathbb{R},$$

which holds for any density generator h and values of ρ in [0, 1). In particular, for $\rho=0$ we have from this last relation that

$$F_{n-1}(z\Delta\mathbf{e}_i; \mathbf{0}, \Delta\{I_n-\mathbf{e}_i\mathbf{e}_i^T\}\Delta^T, h^{(n-1)}_{z^2})=\frac{F_{n-1}(z\mathbf{J}_{n-1}; \mathbf{0}, I_{n-1}, h^{(n-1)}_{z^2})}{(r-1)!(n-r)!}, \quad z \in \mathbb{R}.$$

5. Applications

5.1. Genetic selections

We revisit the application to genetic selection in agricultural research described in the Introduction. We assume that $\mathbf{X} = (X_1, \ldots, X_n)^T$ has an exchangeable multivariate normal distribution. Algorithms for numerical evaluation of multivariate normal cdfs have been studied by Genz [13] and made available in the library `mvtnorm` of the statistical software `R` [22]. We make use of this computing power to study the shape of the distribution of two L-statistics.

First, we consider the exact pdf (14) of the sum of the two best $X_{(n-1)} + X_{(n)}$, related to the selection differential, for a sample of size $n = 5$. Fig. 1 depicts this pdf based on an exchangeable standard multivariate normal distribution with correlation $\rho = 0, 0.1, \ldots, 0.9$. The bold curve is the pdf for $\rho = 0$. The dashed curve is the marginal pdf of the sample. The limiting case $\rho = 1$ is a $N(0, 4)$ pdf. We can see that stronger correlation (larger ρ) among the n animals leads to weaker skewness in the pdf of the sum of the two best animals to be selected for breeding.

Second, we consider the exact pdf (15) of the range $X_{(n)} - X_{(1)}$ for a sample of size $n = 5$. Fig. 2 depicts this pdf based on an exchangeable standard multivariate normal distribution with correlation $\rho = 0, 0.1, \ldots, 0.9$. The bold curve is the pdf for $\rho = 0$. The dashed curve is the marginal pdf of the sample. The limiting case $\rho = 1$ is a degenerate $N(0, 0)$ pdf.

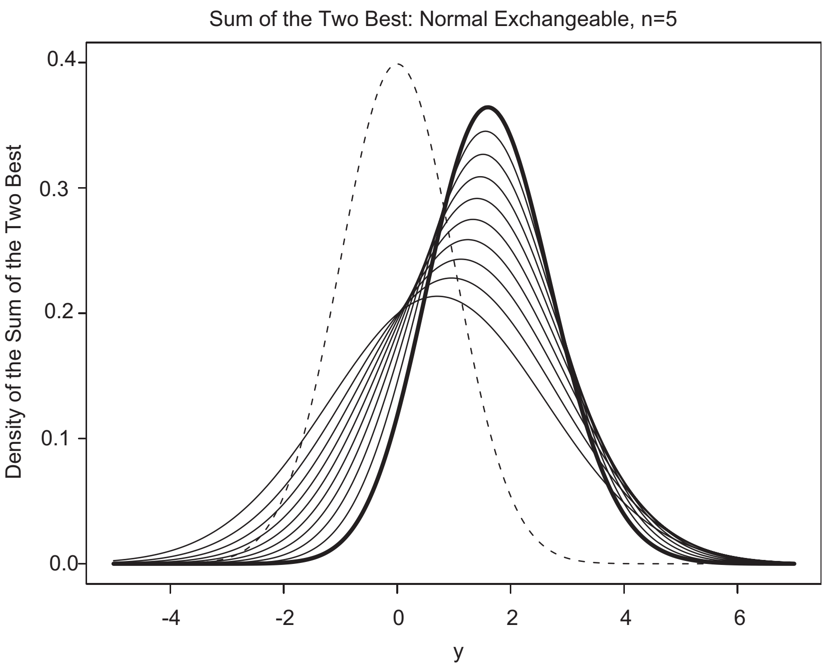

Fig. 1. Genetic selections: exact pdf of the sum of the two best $X_{(n-1)} + X_{(n)}$ for a sample of size $n = 5$ based on an exchangeable standard multivariate normal distribution with correlation $\rho = 0, 0.1, \ldots, 0.9$. The bold curve is the pdf for $\rho = 0$. The dashed curve is the marginal pdf of the sample. The limiting case $\rho = 1$ is a $N(0, 4)$ pdf.

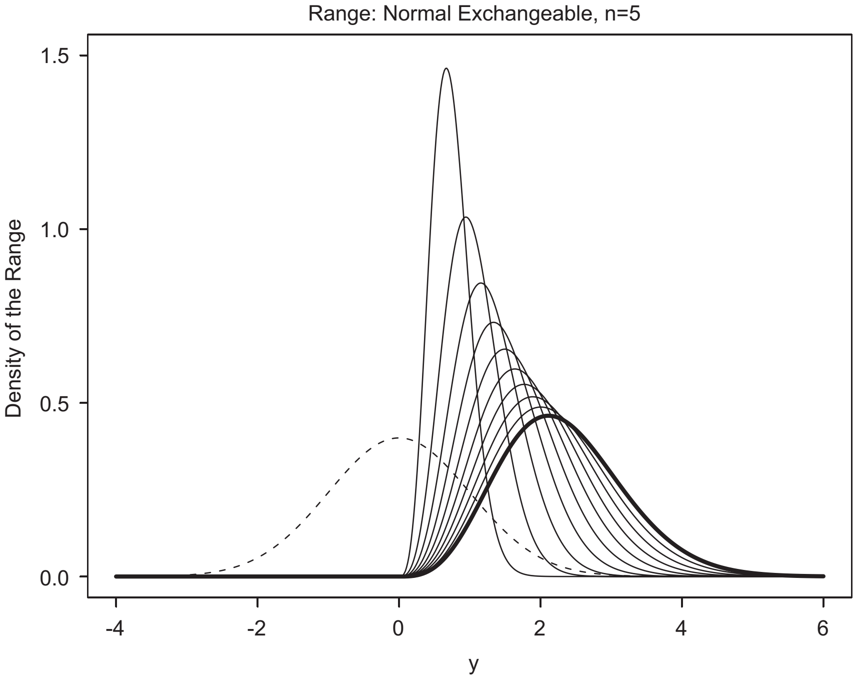

Fig. 2. Genetic selections: exact pdf of the range $X_{(n)} - X_{(1)}$ for a sample of size $n = 5$ based on an exchangeable standard multivariate normal distribution with correlation $\rho = 0, 0.1, \ldots, 0.9$. The bold curve is the pdf for $\rho = 0$. The dashed curve is the marginal pdf of the sample. The limiting case $\rho = 1$ is a degenerate $N(0, 0)$ pdf.

5.2. *Design of cellular phone receivers*

We study the exact pdf of the L-statistic $X_{(n-1)} + X_{(n)}$ representing the two best signals out of $n = 5$ received signals $X_1, \ldots, X_n$ that will be selected by future cellular phones. We assume a multivariate Student t joint distribution for the signal vector $\mathbf{X} = (X_1, \ldots, X_n)^T$, with mean vector $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)^T$, $\mu_i = i$, scale matrix $\Sigma = \sigma^2 I_n$, and degrees of freedom v.

The density generator of the multivariate Student t distribution with v degrees of freedom is $h^{(n)}(v) = c(n, v)v^{v/2}\{v + v\}^{-(n+v)/2}$, $v \geqslant 0$, with $c(n, v) = \Gamma[(n + v)/2]/(\Gamma[v/2]\pi^{n/2})$ and the conditional density generator is $h_a^{(n-1)}(v) = c(n - 1, v + 1)(v + a)^{(v+1)/2}\{v + a + v\}^{-(n+v)/2}$. Denote this distribution by $Student_n(\boldsymbol{\mu}, \Sigma, v)$ with pdf $t_n(\mathbf{x}; \boldsymbol{\mu}, \Sigma, v)$ and cdf $T_n(\mathbf{x}; \boldsymbol{\mu}, \Sigma, v)$, where $\boldsymbol{\mu}$ and Σ are the location vector and dispersion matrix, respectively. Based on (7), the exact pdf of $X_{(n-1)} + X_{(n)}$ is given by

$$2(n-2)! \sum_{i=1}^{n(n-1)/2} t_1(y; \mu_{i_{n-1}} + \mu_{i_n}, 2\sigma^2, v) \times T_{n-1}\left(\Delta \mathbf{a} z_i/\sqrt{2}; \mathbf{0}, \frac{v + z_i^2}{v+1}\Delta(I_n - \mathbf{a}\mathbf{a}^T/2)\Delta^T, v+1\right), \quad y \in \mathbb{R}, \tag{24}$$

where $z_i = (y - \mu_{i_{n-1}} - \mu_{i_n})/(\sqrt{2}\sigma)$ and $\mathbf{a} = (0, 0, 0, 1, 1)^T$.

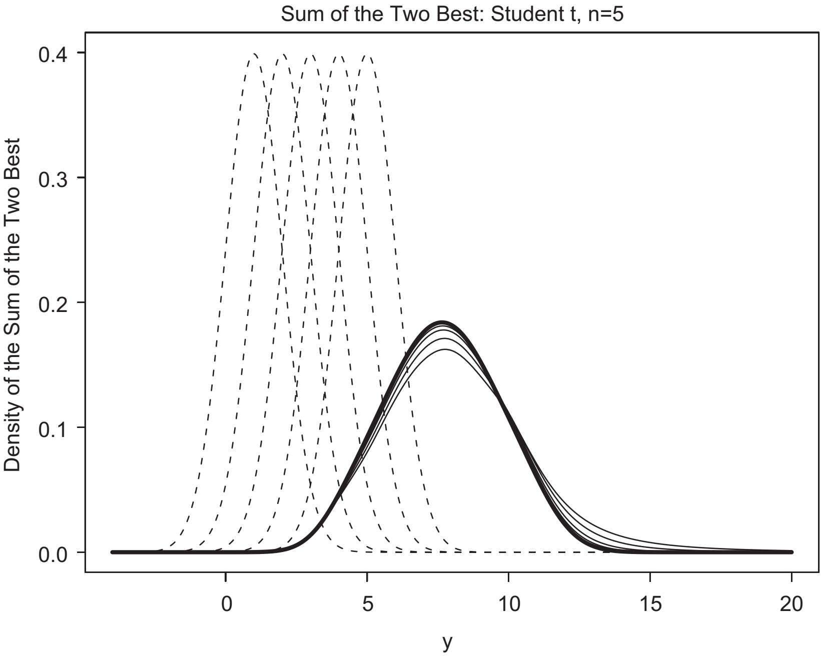

Fig. 3. Design of cellular phone receivers: exact pdf of the sum of the two best $X_{(n-1)} + X_{(n)}$ for a sample of size $n = 5$ from a multivariate Student t distribution with $\nu = 3, 5, 10, 20, \infty$ degrees of freedom. The bold curve is the pdf for $\nu = \infty$. The dashed curves are the five marginal pdfs for the case $\nu = \infty$.

Algorithms for numerical evaluation of the multivariate Student t cdf have been studied by Genz and Bretz [14] and also made available in the statistical software R. Fig. 3 depicts the pdf (24) for $n = 5$ and $\nu = 3, 5, 10, 20, \infty$ degrees of freedom. The bold curve is the pdf for $\nu = \infty$ (the normal case) also derived by Wiens et al. [27] from another approach. The dashed curves are the five marginal pdfs for the case $\nu = \infty$. We can see that heavy tails (small ν) in the marginal pdfs of the received signals lead to stronger skewness in the pdf of the sum of the two best selected signals.

5.3. Visual acuity

We revisit the application to visual acuity described in the Introduction. Let $L\mathbf{X}_{(2)} = a_1 X_{(1)} + a_2 X_{(2)}$, where we assume that $\mathbf{X} = (X_1, X_2)^T$ has a bivariate exchangeable normal distribution $N_2(\mu\mathbf{1}_2, \sigma^2\{I_2 + \gamma\mathbf{1}_2\mathbf{1}_2^T\})$, with $\gamma = \rho/(1-\rho)$ and $\rho \in (-1, 1)$. It is straightforward to obtain the pdf of $L\mathbf{X}_{(2)}$ from (13). In particular, when $a_1 = \alpha$, $a_2 = 1 - \alpha$, $0 \leqslant \alpha \leqslant 1$, we have

$$f_{L\mathbf{X}_{(2)}}(y) = (2/\tau)\phi_1(z)\Phi_1(\lambda z), \quad y \in \mathbb{R}, \tag{25}$$

where $z = (y - \mu)/\tau$, $\tau = \sigma\sqrt{1 - 2\alpha(1-\alpha)(1-\rho)}$ and $\lambda = (1 - 2\alpha)\sqrt{(1-\rho)/(1+\rho)}$. Note that (25) has exactly the form of a skew-normal pdf; see Azzalini [5], who provides all the moments for this model. In particular, $\mathrm{E}(L\mathbf{X}_{(2)}) = \sqrt{2/\pi}\delta\tau^2$ and $\mathrm{Var}(L\mathbf{X}_{(2)}) = \tau^2\{1 - (2/\pi)\delta^2\}$,

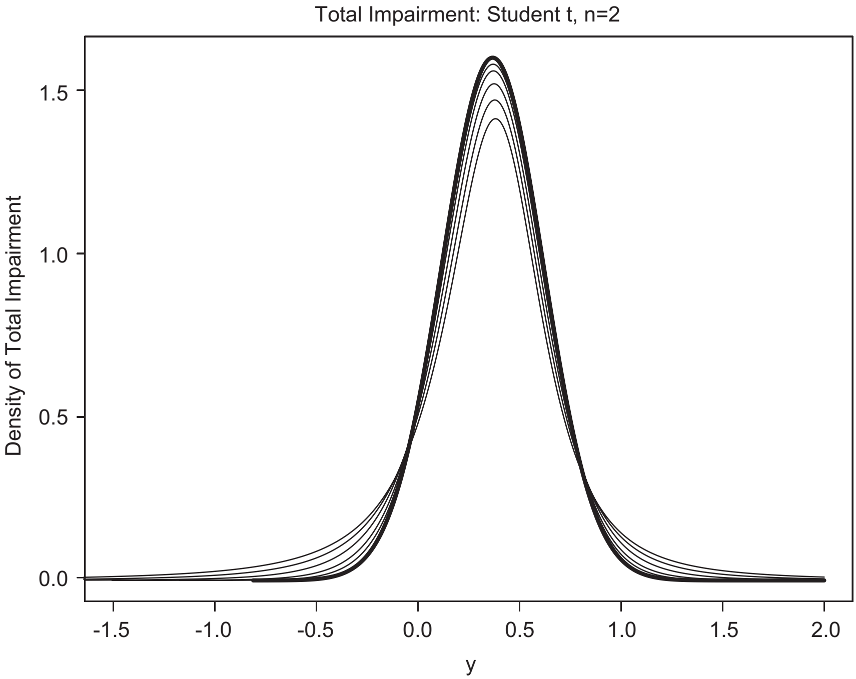

Fig. 4. Visual acuity: exact pdf of the total impairment *TI* for a bivariate Student t distribution with $v = 2, 3, 5, 10, 20, \infty$ degrees of freedom. The bold curve is the pdf for $v = \infty$.

where $\delta = \lambda/\sqrt{1+\lambda^2}$. The special case of the visual impairment is obtained by letting $\alpha = \frac{3}{4}$. We explore the effect of heavy-tailed distributions on the visual impairment *TI* by assuming that $\mathbf{X}$ has a bivariate exchangeable Student t distribution with v degrees of freedom and making use of the exact distribution given by (13). We use the estimates $\hat{\mu}=0.424$, $\hat{\sigma}=0.386$ and $\hat{\rho}=0.496$ obtained by Viana [26] on a real data set. Fig. 4 depicts the exact pdf (13) of *TI* for $v = 2, 3, 5, 10, 20, \infty$ degrees of freedom. The bold curve is the pdf for $v = \infty$ (the normal case), i.e. (25) with $\alpha = \frac{3}{4}$. We can see that heavy tails (small v) in the bivariate distribution of $\mathbf{X}$ leads to stronger skewness in the pdf of the total impairment *TI*.

Acknowledgments

The authors thank two anonymous referees for suggestions that improved the paper. This research was partially supported by Grant FONDECYT 1040865/7050091-Chile and by NSF Grant DMS-0504896.

References

[1] B. Afonja, The moments of the maximum of correlated normal and t-variates, J. Roy. Statist. Soc. Ser. B 34 (1972) 251–262.
[2] R.B. Arellano-Valle, M.D. Branco, M.G. Genton, A unified view on skewed distributions arising from selections, Canad. J. Statist. 34 (2006) 581–601.
[3] R.B. Arellano-Valle, M.G. Genton, On fundamental skew distributions, J. Multivariate Anal. 96 (2005) 93–116.

[4] R.B. Arellano-Valle, M.G. Genton, On the exact distribution of the maximum of dependent random variables, preprint (2006).
[5] A. Azzalini, A class of distributions which includes the normal ones, Scand. J. Statist. 12 (1985) 171–178.
[6] M. Cain, The moment-generating function of the minimum of bivariate normal random variables, Amer. Statist. 48 (1994) 124–125.
[7] M. Cain, M. Pan, Moments of the minimum of bivariate normal random variables, Math. Sci. 20 (1995) 119–122.
[8] S. Cambanis, S. Huang, G. Simons, On the theory of elliptically contoured distributions, J. Multivariate Anal. 11 (1981) 368–385.
[9] C. Crocetta, N. Loperfido, The exact sampling distribution of L-statistics, Metron 63 (2005) 1–11.
[10] H.A. David, N.H. Nagaraja, Order Statistics, third ed., Wiley, New York, 2003 458pp.
[11] K.-T. Fang, S. Kotz, K.-W. Ng, Symmetric Multivariate and Related Distributions. Monographs on Statistics and Applied Probability, vol. 36, Chapman & Hall, Ltd., London, 1990 220pp.
[12] M.G. Genton, Skew-Elliptical Distributions and Their Applications: A Journey Beyond Normality, Edited Volume, Chapman & Hall, CRC, Boca Raton, FL, 2004 416pp.
[13] A. Genz, Numerical computation of multivariate normal probabilities, J. Comput. Graph. Statist. 1 (1992) 141–149.
[14] A. Genz, F. Bretz, Methods for the computation of multivariate t-probabilities, J. Comput. Graph. Statist. 11 (2002) 950–971.
[15] S.S. Gupta, K.C.S. Pillai, On linear functions of ordered correlated normal random variables, Biometrika 52 (1965) 367–379.
[16] W.G. Hill, Order statistics of correlated variables and implications in genetic selection programmes, Biometrics 32 (1976) 889–902.
[17] W.G. Hill, Order statistics of correlated variables and implications in genetic selection programmes. II. Response to selection, Biometrics 33 (1977) 703–712.
[18] N. Loperfido, Statistical implications of selectively reported inferential results, Statist. Probab. Lett. 56 (2002) 13–22.
[19] C. Ma, J. Robinson, Saddlepoint approximations for the difference of order statistics and studentized sample quantiles, J. Roy. Statist. Soc. Ser. B 61 (1999) 563–577.
[20] N.H. Nagaraja, A note on linear functions of ordered correlated normal random variables, Biometrika 69 (1982) 284–285.
[21] I. Olkin, M. Viana, Correlation analysis of extreme observations from a multivariate normal distribution, J. Amer. Statist. Assoc. 90 (1995) 1373–1379.
[22] R Development Core Team, R: a language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria, 2004. ISBN 3-900051-07-0, URL ⟨http://www.R-project.org⟩.
[23] J.O. Rawlings, Order statistics for a special class of unequally correlated multinormal variates, Biometrics 32 (1976) 875–887.
[24] C. Roberts, A correlation model useful in the study of twins, J. Amer. Statist. Assoc. 61 (1966) 1184–1190.
[25] Y.L. Tong, The Multivariate Normal Distribution, Springer, New York, 1990 126pp.
[26] M. Viana, Linear combinations of ordered symmetric observations with applications to visual acuity, in: N. Balakrishnan, C.R. Rao (Eds.), Order Statistics: Applications, Handbook of Statistics, vol. 17, North-Holland, Amsterdam, 1998, pp. 513–524.
[27] D.P. Wiens, N.C. Beaulieu, P. Loskot, On the exact distribution of the sum of the largest $n-k$ out of n normal random variables with differing means, Statistics 40 (2005) 165–173.