Journal of Multivariate Analysis 102 (2011) 1105–1117

Contents lists available at ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

# Characteristic functions of scale mixtures of multivariate skew-normal distributions

Hyoung-Moon Kim [a], Marc G. Genton [b,*]

[a] *Department of Applied Statistics, Konkuk University, Seoul, Republic of Korea*
[b] *Department of Statistics, Texas A&M University, College Station, USA*

ARTICLE INFO

*Article history:*
Received 16 June 2010
Available online 12 April 2011

*AMS subject classifications:*
60E10
62E10

*Keywords:*
Characteristic function
Mixing
Multivariate
Scale mixture
Selection
Skew-normal
Skew-*t*

ABSTRACT

We obtain the characteristic function of scale mixtures of skew-normal distributions both in the univariate and multivariate cases. The derivation uses the simple stochastic relationship between skew-normal distributions and scale mixtures of skew-normal distributions. In particular, we describe the characteristic function of skew-normal, skew-*t*, and other related distributions.



## 1. Introduction

Characteristic functions play an important role in statistics. It is well-known that every bounded and measurable function is integrable with respect to any distribution over the real line. This assures the existence of the characteristic function for any distribution function. Specifically, the characteristic function of a random variable $X$ is the function $\Psi_X : \mathbb{R} \to \mathbb{C}$ defined as

$$\Psi_X(t) = \mathrm{E}\{\exp(\mathrm{i}tX)\}, \quad t \in \mathbb{R}, \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$. An important result is that there is a one-to-one correspondence between the distribution function of a random variable and its characteristic function. This leads to derivations of distributions of many statistics, for example of sums of independent random variables. Moreover, by the Cramér–Wold theorem, a multivariate distribution is uniquely determined by the distributions of all linear combinations of the component variables. By the continuity theorem, the one-to-one correspondence between distribution functions and characteristic functions is continuous. Furthermore, characteristic functions have many pleasant properties including that every characteristic function is uniformly continuous on the whole real line. More properties of characteristic functions can be found in [21,22,12].

In recent years, there has been a growing interest in the construction of parametric classes of non-normal distributions. For instance, the univariate skew-normal distribution has been developed by Azzalini [6] and then further extended to

* Corresponding author.
*E-mail address:* genton@stat.tamu.edu (M.G. Genton).

0047-259X/$ – see front matter 
doi:10.1016/j.jmva.2011.03.004

the multivariate setting by Azzalini and Dalla Valle [10] and Azzalini and Capitanio [8]. Scale mixtures of skew-normal distributions have appeared in [13] and include (skew-) normal distributions as special cases. They have been further extended to skew-elliptical distributions by many authors, see for example [15] and references therein. All these skewed distributions can be cast in the framework of selection distributions that arise under various selection mechanisms, see [3].

The moment generating function has been studied by Arellano-Valle and Genton [4] for fundamental skew-symmetric distributions, by Arellano-Valle and Azzalini [2] for unified skew-normal distributions, by Arellano-Valle et al. [3] for scale mixtures of skew-normal distributions, and by Arellano-Valle and Genton [5] for quadratic forms in random vectors with a selection distribution. Although characteristic functions have nice properties and determine the distribution function uniquely, there have been surprisingly few research articles on the characteristic functions of the aforementioned skewed distributions. Pewsey [24] developed the characteristic function of the univariate skew-normal distribution. Kozubowski and Podgórski [19] discussed the origins and inter-relations of various skew-Laplace distributions. In doing so, they presented the characteristic functions of some skew-Laplace distributions. Therefore, there is a need to develop characteristic functions of these skewed distributions, both in the univariate and multivariate cases. We concentrate on the characteristic functions of scale mixtures of multivariate skew-normal distributions which include some well-known distributions, for example such as the skew-normal, skew-$t$, and skew-slash [27] distributions among others.

This paper is organized as follows. In Section 2, we review scale mixtures of multivariate skew-normal distributions and their properties. In Section 3, we first derive the characteristic function of the univariate skew-normal distribution and prove that the derived formula is the same as the one given by Pewsey [24]. We give also another form of that characteristic function and various methods of proof. The characteristic function of the skew-$t$ distribution is obtained using scale mixtures of skew-normal distributions and the Laplace–Stieltjes transform. Using a lemma by Azzalini [7], we derive the characteristic function of scale mixtures of skew-normal distributions in a series representation when the mixing moments exist. Generally, we derive the characteristic function of scale mixtures of skew-normal distributions as mixtures of characteristic functions of skew-normal distributions. In Section 4, the characteristic functions of multivariate skew-normal and skew-$t$ distributions are obtained. Finally, the characteristic function of scale mixtures of multivariate skew-normal distributions is obtained as a mixture of the characteristic functions of multivariate skew-normal distributions.

## 2. Scale mixtures of skew-normal distributions

### 2.1. Univariate case

A random variable $Z$ has a skew-normal distribution developed by Azzalini [6] if its probability density function (pdf) is

$$f_Z(z) = 2\phi(z)\Phi(\alpha z), \quad z \in \mathbb{R}, \tag{2}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ denote the pdf and cumulative distribution function (cdf) of the standard normal $N(0, 1)$ distribution, respectively. The parameter $\alpha \in \mathbb{R}$ controls the skewness (shape) of the distribution. When $Z$ has the pdf (2), we write $Z \sim \text{SN}(\alpha)$. If $\alpha = 0$, then (2) reduces to the $N(0, 1)$ pdf. The location-scale extension of $Z$ is

$$X = \xi + \omega Z, \tag{3}$$

with pdf

$$f_X(x) = 2\phi(x; \xi, \omega^2)\Phi\left(\alpha \frac{x-\xi}{\omega}\right), \quad x \in \mathbb{R}, \tag{4}$$

where $\phi(\cdot; \xi, \omega^2)$ denotes the normal pdf with location $\xi \in \mathbb{R}$ and scale $\omega > 0$. In this case, we denote the skew-normal random variable $X \sim \text{SN}(\xi, \omega^2, \alpha)$, similarly to Azzalini and Capitanio [8].

Scale mixtures of skew-normal distributions appeared in [13] and are defined by the following stochastic representation:

$$Y = \xi + W(\eta)^{1/2} X, \tag{5}$$

where $X$ has the skew-normal distribution $\text{SN}(0, \omega^2, \alpha)$ and $\eta$ is a mixing variable with cdf $H(\eta)$ and a weight function $W(\eta)$, independent of $X$. If $X$ has a normal distribution, i.e. $X \sim N(0, \omega^2)$, then (5) is the stochastic representation of scale mixtures of normal distributions.

Consequently, the pdf of $Y$ is

$$f_Y(y) = 2\int_0^\infty \phi(y; \xi, W(\eta)\omega^2)\Phi\left(\alpha \frac{y-\xi}{W(\eta)^{1/2}\omega}\right) \mathrm{d}H(\eta) \tag{6}$$

where $H(\eta)$ is the cdf of $\eta$. One particular case of this distribution is the skew-normal distribution, for which $H$ is degenerate, with $W(\eta) = 1$. In this case the pdf (6) reduces to (4). We summarize special cases of scale mixtures of skew-normal distributions in Table 1 along with their mixing pdf $h(\eta)$. These mixing distributions also apply to multivariate

**Table 1**
Special cases of scale mixtures of skew-normal distributions.

| Distribution | $W(\eta)$ | $h(\eta)$ |
|---|---|---|
| Skew-$t$ | $1/\eta$ | $\frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}\eta^{\nu/2-1}e^{-\nu\eta/2},\ \eta > 0$ |
| Skew-logistic | $4\eta^2$ | $8\sum_{k=1}^{\infty}(-1)^{k+1}k^2\eta\ \exp(-2k^2\eta^2),\ \eta > 0$ |
| Skew-slash | $1/\eta^{2/q},\ q > 0$ | $\eta \sim U(0, 1)$ |

situations. Furthermore, when the distribution $H$ is a discrete measure on $\{\eta_1, \eta_2, \ldots, \eta_n\}$ with probabilities $p_1, p_2, \ldots, p_n$, respectively, then the pdf of the finite mixture of skew-normal distributions [13] is

$$f_Y(y) = 2\sum_{j=1}^{n} p_j\phi(y; \xi, W(\eta_j)\omega^2)\Phi\left(\alpha\frac{y-\xi}{W(\eta_j)^{1/2}\omega}\right), \tag{7}$$

where $0 \le p_j \le 1$ and $\sum_{j=1}^{n} p_j = 1$. One special case happens when $W(\eta) = 1/\eta$ and $H$ is a discrete measure on $\{\eta_1 = \gamma, \eta_2 = 1\}$ with probabilities $p$, $1-p$, respectively. Then the skew-contaminated-normal distribution [20] is obtained with pdf

$$f_Y(y) = 2\left\{p\phi\left(y; \xi, \frac{\omega^2}{\gamma}\right)\Phi\left(\alpha\frac{y-\xi}{\omega/\gamma^{1/2}}\right) + (1-p)\phi(y; \xi, \omega^2)\Phi\left(\alpha\frac{y-\xi}{\omega}\right)\right\},$$

where $0 < p < 1$ and $0 < \gamma \le 1$.

### 2.2. Multivariate case

The multivariate version of the skew-normal distribution has been introduced by Azzalini and Dalla Valle [10] and Azzalini and Capitanio [8]. Recent developments related to multivariate skew-normal distributions are summarized well in [15,7,2]. A $k$-dimensional random vector $Z$ is said to have a multivariate skew-normal distribution if it is continuous with pdf

$$f_Z(z) = 2\phi_k(z; \Omega_z)\Phi(\alpha^T z), \quad z \in \mathbb{R}^k,$$

where $\phi_k(z; \Omega_z)$ is the $k$-dimensional normal pdf with zero mean and correlation matrix $\Omega_z$, and $\alpha$ is a $k$-dimensional vector controlling skewness (shape). We shall write $Z \sim \mathrm{SN}_k(\Omega_z, \alpha)$. To incorporate location and scale parameters, we write

$$X = \xi + \omega Z, \tag{8}$$

where $\xi = (\xi_1, \ldots, \xi_k)^T$, $\omega = \mathrm{diag}(\omega_1, \ldots, \omega_k)$, $\omega_i = \sqrt{\omega_{ii}}$, and $\Omega = (\omega_{ij})$ is a full rank $k \times k$ covariance matrix. Then the $k$-dimensional random vector $X$ is said to have a multivariate skew-normal distribution with pdf

$$f_X(x) = 2\phi_k(x-\xi; \Omega)\Phi\{\alpha^T\omega^{-1}(x-\xi)\}, \quad x \in \mathbb{R}^k, \tag{9}$$

where $\Omega = \omega\Omega_z\omega$. We shall write $X \sim \mathrm{SN}_k(\xi, \Omega, \alpha)$.

There is a multivariate stochastic representation of the skew-normal distribution. Assume that $V_0 \sim N(0, 1)$ and $V_1 \sim N_k(0, R)$ are independent random variables, where $R$ is a correlation matrix. Let $\Delta = \mathrm{diag}(\delta_1, \ldots, \delta_k)$ where $\delta_j \in (-1, 1)$ for all $j$s. Then

$$Z = \Delta 1_k|V_0| + (I_k - \Delta^2)^{1/2}V_1 \tag{10}$$

has a $\mathrm{SN}_k(\Omega_z, \alpha)$ distribution, with a known relationship between the $(R, \Delta)$ and the $(\Omega_z, \alpha)$ set of parameters [7,10].

Scale mixtures of multivariate skew-normal distributions are related to the multivariate skew-normal distribution by the following stochastic equation:

$$Y = \xi + W(\eta)^{1/2}X, \tag{11}$$

where $X$ has the multivariate skew-normal distribution $\mathrm{SN}_k(0, \Omega, \alpha)$, and $\eta$ is a mixing variable with cdf $H(\eta)$ and a weight function $W(\eta)$, independent of $X$. If $X$ has multivariate normal distribution, i.e. $X \sim N_k(0, \Omega)$, then (11) is the stochastic representation for scale mixtures of multivariate normal distributions.

Therefore the pdf of $Y$ is

$$f_Y(y) = 2\int_0^{\infty}\phi_k(y; \xi, W(\eta)\Omega)\Phi\left\{W(\eta)^{-1/2}\alpha^T\omega^{-1}(y-\xi)\right\}\mathrm{d}H(\eta).$$

One particular case of this distribution is the multivariate skew-normal distribution, for which $H$ is degenerate, with $W(\eta) = 1$. In this case the pdf is given by (9). When the distribution $H$ is a discrete measure on $\{\eta_1, \eta_2, \ldots, \eta_n\}$ with probabilities $p_1, p_2, \ldots, p_n$, respectively, then the pdf of the finite mixture of multivariate skew-normal distributions is

$$f_Y(y) = 2\sum_{j=1}^{n} p_j\phi_k(y; \xi, W(\eta_j)\Omega)\Phi\left\{W(\eta_j)^{-1/2}\alpha^T\omega^{-1}(y-\xi)\right\}, \tag{12}$$

where $0 \le p_j \le 1$ and $\sum_{j=1}^{n} p_j = 1$.

Another special case happens when $W(\eta) = 1/\eta$ and $H$ is a discrete measure on $\{\eta_1 = \gamma, \eta_2 = 1\}$ with probabilities $p, 1-p$, respectively, then the pdf of the multivariate skew-contaminated-normal distribution is

$$f_Y(y) = 2\left[p\phi_k(y;\xi,\gamma^{-1}\Omega)\Phi\left\{\gamma^{1/2}\alpha^T\omega^{-1}(y-\xi)\right\} + (1-p)\phi_k(y;\xi,\Omega)\Phi\left\{\alpha^T\omega^{-1}(y-\xi)\right\}\right],$$

where $0 < p < 1$ and $0 < \gamma \le 1$.

Other scale mixtures of multivariate skew-normal distributions can be found in [13]. For example, the multivariate skew-$t$ distribution; see also [9,11]. Branco and Dey [13] described multivariate skew-logistic distributions, whereas Wang and Genton [27] described multivariate and skew-multivariate extensions of the slash distribution.

## 3. Characteristic functions: univariate case

### 3.1. Skew-normal distributions

The characteristic function (1) of the univariate skew-normal distribution has been derived by Pewsey [24]:

$$\Psi_X(t) = \exp\left(\mathrm{i}\xi t - \omega^2 t^2/2\right)\{1 + \mathrm{i}\tau(\delta\omega t)\}, \quad t \in \mathbb{R}, \tag{13}$$

where $X \sim \mathrm{SN}(\xi, \omega^2, \alpha)$, $\delta = \alpha/\sqrt{1+\alpha^2}$, and

$$\tau(x) = \int_0^x \sqrt{2/\pi}\exp(u^2/2)\mathrm{d}u, \quad x > 0, \tag{14}$$

and $\tau(-x) = -\tau(x)$. He used a differential equations approach to obtain this result.

Here we derive the characteristic function of the skew-normal distribution based on a simple fact. If a random variable admits a pdf, then the characteristic function is its dual, in the sense that each of them is a Fourier transform of the other. In other words, since the moment generating function $M_X(t)$ of the skew-normal distribution exists for all $t \in \mathbb{R}$, we can use $\Psi_X(t) = M_X(\mathrm{i}t)$ as long as $M_X(t)$ has an explicit formula, which is guaranteed by the Lemma 1 below. However, the characteristic function of a distribution always exists, even when the pdf or moment generating function do not.

**Lemma 1.** *If the function $\Phi(z)$, $z = x + \mathrm{i}y$, is defined by integration along a path in the complex plane parallel with the x-axis from $-\infty + \mathrm{i}y$ to $x + \mathrm{i}y$, then:*

(i) *$\Phi(z)$ is convergent;*
(ii) *$\Phi(z) = \Phi(x+\mathrm{i}y) = \exp\left(\frac{y^2}{2}\right)\int_{-\infty}^{x}\exp(-\mathrm{i}ty)\phi(t)\mathrm{d}t$, where $\phi(t)$ is the standard normal pdf;*
(iii) *When $x = 0$:*

$$\Phi(\mathrm{i}y) = \frac{1}{2} + \frac{\mathrm{i}}{\sqrt{\pi}}\int_0^{y/\sqrt{2}}\exp(t^2)\mathrm{d}t. \tag{15}$$

**Proof.** Parts (i) and (ii) appeared in [26] so we only need to prove (15) in part (iii). We know that $\exp(-\mathrm{i}ty) = \cos(ty) - \mathrm{i}\sin(ty)$. Hence

$$\int_{-\infty}^{0}\cos(ty)\phi(t)\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\cos(ty)\exp(-t^2/2)\mathrm{d}t.$$

Letting $t = -2z$ and then using the integral formula 7.4.6 [1], we have

$$\int_{-\infty}^{0}\cos(ty)\phi(t)\mathrm{d}t = \frac{1}{2}\exp(-y^2/2),$$

$$\int_{-\infty}^{0}\sin(ty)\phi(t)\mathrm{d}t = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0}\sin(ty)\exp(-t^2/2)\mathrm{d}t,$$

whereas using the integral formula 7.4.7 [1], we have

$$\int_{-\infty}^{0}\sin(ty)\phi(t)\mathrm{d}t = -\frac{1}{\sqrt{\pi}}\exp(-y^2/2)\int_0^{y/\sqrt{2}}\exp(t^2)\mathrm{d}t.$$

Hence the result follows. □

Nelson [23] gave an analytic approximation to the complex normal probability integral, $\Phi(x + \mathrm{i}y)$. Therefore the characteristic function of the skew-normal distribution is as follows.

**Theorem 1.** *Let $X$ have a univariate skew-normal distribution, i.e. $X \sim \mathrm{SN}(\xi, \omega^2, \alpha)$. Then the characteristic function of $X$ is:*

$$\Psi_X(t) = 2\exp\left(\mathrm{i}\xi t - \omega^2 t^2/2\right)\Phi(\mathrm{i}\delta\omega t).$$

*Furthermore, $\Psi_X(t)$ is also equal to (13).*

**Proof.** We provide three methods to prove this theorem.

*Method* 1: Since $M_X(s) = \mathrm{E}\{\exp(sX)\} = 2\exp\left(\xi s + \omega^2 s^2/2\right)\Phi(\delta\omega s)$, $s = x + \mathrm{i}y$, $x, y \in \mathbb{R}$, is an analytic continuation of the skew-normal moment generating function, we have that $\Psi_X(t) = \mathrm{E}\{\exp(\mathrm{i}tX)\} = M_X(\mathrm{i}t)$. By Lemma 1,

$$\Psi_X(t) = \exp\left(\mathrm{i}\xi t - \omega^2 t^2/2\right)\left(1 + \frac{2\mathrm{i}}{\sqrt{\pi}}\int_0^{\delta\omega t/\sqrt{2}} \exp(u^2)\mathrm{d}u\right).$$

Letting $u = \nu/\sqrt{2}$, the result follows.

*Method* 2: When $Z \sim \mathrm{SN}(\alpha)$, Henze [16] gave this simple stochastic representation:

$$Z = \delta|V_0| + \sqrt{1-\delta^2}V_1,$$

where $V_0$ and $V_1$ follow independent standard normal distributions, respectively. Therefore

$$\Psi_Z(t) = \mathrm{E}\{\exp(\mathrm{i}t\delta|V_0|)\}\mathrm{E}\{\exp(\mathrm{i}t\sqrt{1-\delta^2}V_1)\}$$

by independence of $V_0$ and $V_1$. We have:

$$\begin{aligned}
\mathrm{E}\{\exp(\mathrm{i}t\sqrt{1-\delta^2}V_1)\} &= \int_{-\infty}^{\infty} \exp(\mathrm{i}t\sqrt{1-\delta^2}v_1)\frac{1}{\sqrt{2\pi}}\exp(-v_1^2/2)\mathrm{d}v_1 \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\{-(v_1^2 - 2\mathrm{i}t\sqrt{1-\delta^2}v_1)/2\}\mathrm{d}v_1 \\
&= \exp\{-t^2(1-\delta^2)/2\}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\exp\{-(v_1 - \mathrm{i}t\sqrt{1-\delta^2})^2/2\}\mathrm{d}v_1 \\
&= \exp\{-t^2(1-\delta^2)/2\},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{E}\{\exp(\mathrm{i}t\delta|V_0|)\} &= \int_{-\infty}^{\infty} \exp(\mathrm{i}t\delta|v_0|)\frac{1}{\sqrt{2\pi}}\exp(-v_0^2/2)\mathrm{d}v_0 \\
&= 2\int_{-\infty}^{0} \exp(-\mathrm{i}t\delta v_0)\frac{1}{\sqrt{2\pi}}\exp(-v_0^2/2)\mathrm{d}v_0
\end{aligned}$$

since the integrand is an even function. Hence

$$\mathrm{E}\{\exp(\mathrm{i}t\delta|V_0|)\} = 2\exp\left(-t^2\delta^2/2\right)\Phi(\mathrm{i}\delta t) \tag{16}$$

by Lemma 1. Then

$$\Psi_Z(t) = 2\exp\left(-t^2/2\right)\Phi(\mathrm{i}\delta t).$$

The remaining calculation is similar to Method 1. Using (3) and standard properties of the characteristic function, the result follows.

*Method* 3: We calculate $\mathrm{E}\{\exp(\mathrm{i}t\delta|V_0|)\}$ using complex contour integration. We already know $\mathrm{E}\{\exp(\mathrm{i}t\sqrt{1-\delta^2}V_1)\}$ from Method 2. After simple algebra, we have that

$$\mathrm{E}\{\exp(\mathrm{i}t\delta|V_0|)\} = \sqrt{\frac{2}{\pi}}\exp\left(-t^2\delta^2/2\right)\int_0^{\infty}\exp\left\{-(x - \mathrm{i}t\delta)^2/2\right\}\mathrm{d}x.$$

Let $f(z) = \exp\left(-z^2/2\right)$ defined on the complex domain. Also define the closed contour describing the boundary of the unit box with counterclockwise orientation having four vertices at $(0, -t\delta)$, $(M, -t\delta)$, $(M, 0)$, and $(0, 0)$; see Fig. 1.

Because $f(z)$ is an entire function (i.e. it has no singularity on the whole complex domain), $\oint_R f(z)\mathrm{d}z = 0$ by the Cauchy integral theorem. The closed contour can be divided into four pieces of paths $R_1$, $R_2$, $R_3$, and $R_4$. Hence

$$\oint_{R_1} f(z)\mathrm{d}z = -\left(\oint_{R_2} + \oint_{R_3} + \oint_{R_4}\right)f(z)\mathrm{d}z$$

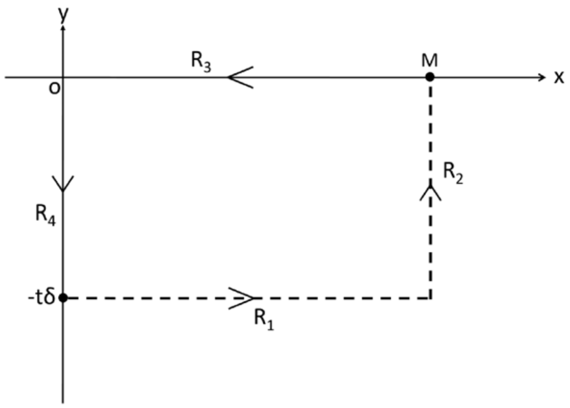


**Fig. 1.** A closed contour $R = R_1 \cup R_2 \cup R_3 \cup R_4$.

and

$$\int_0^\infty \exp\left\{-(x - \mathrm{i}t\delta)^2/2\right\} \mathrm{d}x = \lim_{M\to\infty} \oint_{R_1} \exp\left(-z^2/2\right) \mathrm{d}z \tag{17}$$

since $z = x - \mathrm{i}t\delta$ on $R_1$. Even though $M \to \infty$, (17) holds since $R$ is still a closed contour. Now we want to evaluate each integration. First:

$$\begin{aligned}\oint_{R_2} f(z)\mathrm{d}z &= \mathrm{i}\int_{-t\delta}^0 \exp\left\{-(M + \mathrm{i}y)^2/2\right\} \mathrm{d}y \\ &\le \exp\left(-M^2/2\right) |t\delta| \max_{y\in[-t\delta,0]} \exp\left(y^2/2\right) \longrightarrow 0\end{aligned}$$

as $M \to \infty$. Then:

$$\oint_{R_3} f(z)\mathrm{d}z = -\int_0^M \exp\left(-x^2/2\right) \mathrm{d}x \longrightarrow -\sqrt{\pi/2}$$

as $M \to \infty$. Next:

$$\oint_{R_4} f(z)\mathrm{d}z = -\mathrm{i}\int_{-t\delta}^0 \exp\left(y^2/2\right) \mathrm{d}y = -\mathrm{i}\int_0^{t\delta} \exp\left(x^2/2\right) \mathrm{d}x.$$

Combining each integral, we have

$$\int_0^\infty \exp\left\{-(x - \mathrm{i}t\delta)^2/2\right\} \mathrm{d}x = \sqrt{\frac{\pi}{2}} + \mathrm{i}\int_0^{t\delta} \exp\left(x^2/2\right) \mathrm{d}x.$$

Therefore,

$$\mathrm{E}\{\exp(\mathrm{i}t\delta|V_0|)\} = \exp\left(-t^2\delta^2/2\right)\left[1 + \mathrm{i}\sqrt{\frac{2}{\pi}}\int_0^{t\delta} \exp\left(x^2/2\right) \mathrm{d}x\right]. \tag{18}$$

Hence the result follows after using simple algebra and standard properties of the characteristic function. □

Here (16) and (18) are the same so we proved (15) in a direct way using contour integration. Furthermore, when $\delta = 1$, (16) and (18) are the characteristic function of the standard half-normal distribution or the standard truncated normal distribution with the left truncation point at 0. Here standard means that we transform the standard normal distribution to the half-normal distribution. We summarize these results in a corollary.

**Corollary 1.** *Let $Z \sim N(0, 1)$. Then $Y = |Z|$ follows the standard half-normal distribution or the standard truncated normal distribution with the left truncation point at 0, and the characteristic function of $Y$ is:*

$$\Psi_Y(t) = 2\exp\left(-t^2/2\right) \Phi(\mathrm{i}t) = \exp\left(-t^2/2\right)\{1 + \mathrm{i}\tau(t)\},$$

*where $\tau(x)$ is given in (14).*

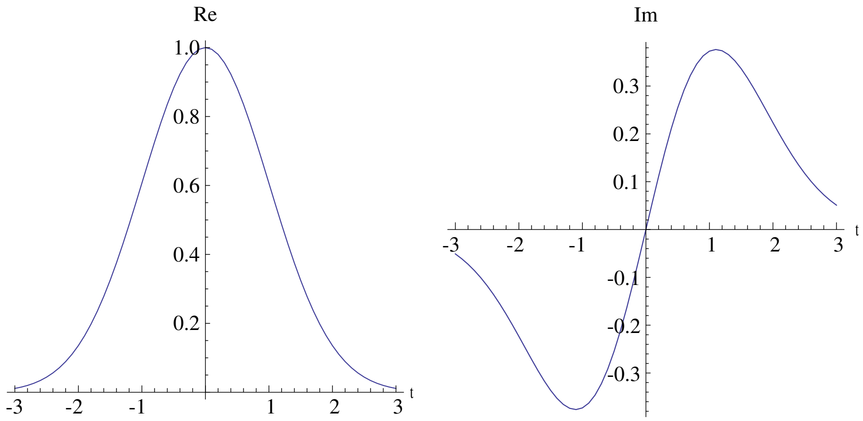


**Fig. 2.** Real (left panel) and imaginary (right panel) parts of the characteristic function $\Psi_X(t)$ of a skew-normal distribution, $X \sim \mathrm{SN}(0, 1, 1)$.

We plot the characteristic function of the skew-normal distribution with parameters $\xi = 0$, $\omega = 1$, and $\alpha = 1$ in Fig. 2. The left (right) panel corresponds to the real (imaginary) part of the characteristic function of the skew-normal distribution. For the standard normal distribution, i.e. when $\alpha = 0$, the imaginary part of the characteristic function is always 0.

The characteristic function of a finite mixture of skew-normal distributions is described next.

**Corollary 2.** *Let Y follow a finite mixture of skew-normal distributions with pdf (7). Then the characteristic function of Y is:*

$$\begin{aligned}\Psi_Y(t) &= 2\sum_{j=1}^{n} p_j \exp\left(\mathrm{i}\xi t - W(\eta_j)\omega^2 t^2/2\right) \Phi(\mathrm{i}\delta\sqrt{W(\eta_j)}\omega t)\\ &= \sum_{j=1}^{n} p_j \exp\left(\mathrm{i}\xi t - W(\eta_j)\omega^2 t^2/2\right)\left\{1 + \mathrm{i}\tau\left(\delta\sqrt{W(\eta_j)}\omega t\right)\right\}.\end{aligned}$$

**Proof.** The proof is similar to that of Theorem 1. □

As a special case of Corollary 2, the characteristic function of the skew-contaminated-normal distribution is:

$$\begin{aligned}\Psi_Y(t) &= 2\exp(\mathrm{i}\xi t)\left[p\exp(-\gamma^{-1}\omega^2 t^2/2)\Phi(\mathrm{i}\delta\gamma^{-1/2}\omega t) + (1-p)\exp(-\omega^2 t^2/2)\Phi(\mathrm{i}\delta\omega t)\right]\\ &= \exp(\mathrm{i}\xi t)\left[p\exp(-\gamma^{-1}\omega^2 t^2/2)\left\{1 + \mathrm{i}\tau\left(\delta\gamma^{-1/2}\omega t\right)\right\} + (1-p)\exp(-\omega^2 t^2/2)\left\{1 + \mathrm{i}\tau\left(\delta\omega t\right)\right\}\right].\end{aligned}$$

### *3.2. Skew-t distributions*

The characteristic function of the skew-*t* distribution is obtained using scale mixtures of skew-normal distributions and the Laplace–Stieltjes transform. From (5), the skew-*t* distribution of a random variable $Y$ is related to the skew-normal distribution by the following stochastic equation:

$$Y = \xi + \eta^{1/2} X, \tag{19}$$

where $X$ has a skew-normal distribution, $X \sim \mathrm{SN}(0, \omega^2, \alpha)$, and $\eta$ has an inverse-Gamma distribution, $\eta \sim \mathrm{IG}(\nu/2, \nu/2)$, i.e. the pdf of $\eta$ is

$$\frac{(\nu/2)^{\nu/2}}{\Gamma(\nu/2)}\eta^{-\nu/2-1}\exp\{-\nu/(2\eta)\}, \quad \eta > 0.$$

A particular case of the skew-*t* distribution is the skew-Cauchy distribution, when $\nu = 1$. Also when $\nu \to \infty$, we obtain the skew-normal distribution as the limiting case. To apply the Laplace–Stieltjes transform, we use (19). For skew-*t* distributions, this stochastic equation is the same as (5) because of the relationship between a Gamma distribution and an inverse-Gamma distribution.

**Theorem 2.** *Let Y follow the skew-t distribution defined by (19). Then the characteristic function of Y is:*

$$\Psi_Y(t) = \exp(\mathrm{i}\xi t)\left\{\Psi_T(\omega t) + \mathrm{i}\tau^*(\delta, \omega t)\right\},$$

**Table 2**
Moments of mixing distributions for special cases of scale mixtures of skew-normal distribution.

| Distribution | $c_m$ |
|---|---|
| Skew-$t$ | $\frac{(\nu/2)^{m/2}\Gamma((\nu-m)/2)}{\Gamma(\nu/2)}$, $\nu > m$ |
| Skew-logistic | $2^{1+m/2}\Gamma(m/2+1)\sum_{k=1}^{\infty}(-1)^{k+1}/k^m$ |
| Skew-slash | $\frac{q}{q-m}$, $q > m$ |
| Finite mixture of skew-normal | $\sum_{j=1}^{n} W(\eta_j)^{m/2}p_j$ |
| Skew-contaminated-normal | $\frac{p}{\gamma^{m/2}} + 1 - p$ |

*where*

$$\Psi_T(t) = \frac{K_{\nu/2}(\sqrt{\nu}|t|)(\sqrt{\nu}|t|)^{\nu/2}}{\Gamma(\nu/2)2^{\nu/2-1}}, \quad \text{for } t \in \mathbb{R},\ \nu > 0,$$

$$\tau^*(\delta, \omega t) = \int_0^\infty \exp(-\eta\omega^2 t^2/2)\tau(\delta\sqrt{\eta}\omega t)\mathrm{d}H(\eta), \quad \text{for } \delta t > 0, \tag{20}$$

*with* $\tau^*(\delta, -\omega t) = -\tau^*(\delta, \omega t)$*, and* $K_\lambda(w)$ *is the integral representation of the modified Bessel function of the third kind*

$$K_\lambda(w) = \frac{1}{2}\int_0^\infty x^{\lambda-1}\exp\left\{-\frac{1}{2}w\left(x+\frac{1}{x}\right)\right\}\mathrm{d}x, \quad w > 0 \text{ for } \lambda \in \mathbb{R}.$$

**Proof.** The conditional distribution of $Y$ given $\eta$ follows a skew-normal distribution, i.e. $Y|\eta \sim \mathrm{SN}(\xi, \eta\omega^2, \alpha)$. Then the characteristic function of $Y$ is

$$\begin{aligned}\Psi_Y(t) &= \int_0^\infty \int_{\mathbb{R}} \exp(\mathrm{i}ty)f(y|\eta)\mathrm{d}y\mathrm{d}H(\eta) = \int_0^\infty \Psi_{Y|\eta}(t)\mathrm{d}H(\eta)\\ &= \exp(\mathrm{i}\xi t)\int_0^\infty \exp\left(-\eta\omega^2t^2/2\right)\left\{1 + \mathrm{i}\tau(\delta\sqrt{\eta}\omega t)\right\}\mathrm{d}H(\eta)\\ &= \exp(\mathrm{i}\xi t)\left\{L_\eta\left(\omega^2t^2/2\right) + \mathrm{i}\int_0^\infty \exp\left(-\eta\omega^2t^2/2\right)\tau(\delta\sqrt{\eta}\omega t)\mathrm{d}H(\eta)\right\},\end{aligned}$$

where $L_\eta(\gamma)$ is the Laplace–Stieltjes transform

$$L_\eta(\gamma) = \mathrm{E}\{\exp(-\gamma\eta)\} = \int_0^\infty \exp(-\gamma\eta)\mathrm{d}H(\eta)$$

when $\eta$ is a non-negative random variable. So $L_\eta\left(\omega^2t^2/2\right)$ becomes the characteristic function of a Student's $t$ distribution, $\Psi_T(\omega t)$, after developing the characteristic function of the symmetric generalized hyperbolic distribution and then using some properties of the modified Bessel function of the third kind [17]. The remaining calculation is obvious so the result follows. Here $\Psi_T(t)$ is the characteristic function of a Student's $t$ distribution with degrees of freedom $\nu$ and the integrand of (20) without a constant $2/\sqrt{\pi}$ becomes Dawson's integral when $\delta = 1$. □

### 3.3. *Scale mixtures of skew-normal distributions*

Branco and Dey [13] developed scale mixtures of skew-normal distributions. The moments of mixing distributions are defined by:

$$\mathrm{E}\left(W(\eta)^{m/2}\right) = c_m, \quad \text{where } m = 1, 2, \ldots.$$

We summarize the moments of mixing distributions for some special cases of scale mixtures of skew-normal distributions in Table 2. There are other scale mixtures of skew-normal distributions, for example, skew-stable distributions. However, the higher moments of the latter do not exist. Indeed, the mixing distribution is a positive stable distribution which has all moments of order less than $\alpha \in (0, 2]$, but none greater than $\alpha$, where the $\alpha$ parameter, known as the characteristic exponent, defines the fatness of the tails (large $\alpha$ implies thin tails). However Buckle [14] nicely considered stable distributions in a Bayesian way with MCMC. See [25] for positive stable distributions.

The following lemma [7] is needed for deriving the characteristic function of scale mixtures of skew-normal distributions.

**Lemma 2.** *Let* $Z \sim \mathrm{SN}(\alpha)$*. Then the moments of* $Z$ *are:*

$$E(Z^r) = \begin{cases} 1 \times 3 \times \cdots \times (r-1) & \text{if } r \text{ is even},\\ \dfrac{\sqrt{2}(2k+1)!\alpha}{\sqrt{\pi}(1+\alpha^2)^{k+1/2}2^k}\displaystyle\sum_{m=0}^{k}\frac{m!(2\alpha)^{2m}}{(2m+1)!(k-m)!} & \text{if } r = 2k+1 \text{ and } k = 0, 1, \ldots.\end{cases}$$

**Theorem 3.** *Let $Y$ follow a scale mixture of skew-normal distributions, i.e. $Y = \xi + W(\eta)^{1/2}X$, where $X$ follows a skew-normal distribution, $X \sim \mathrm{SN}(0, \omega^2, \alpha)$, and $\eta$ is a mixing variable with cdf $H(\eta)$ and a weight function $W(\eta)$ independent of $X$. Then the characteristic function of $Y$ is:*

$$\Psi_Y(t) = \exp(\mathrm{i}\xi t)\left[\sum_{n=0}^{\infty} \frac{(-1)^n(\omega t)^{2n}c_{2n}}{(2n)!}\{1 \times 3 \times \cdots \times (2n-1)\}\right.$$
$$\left. + \mathrm{i}\sum_{n=0}^{\infty} \frac{\sqrt{2}(-1)^n(\omega t)^{2n+1}c_{2n+1}\alpha}{\sqrt{\pi}(1+\alpha^2)^{n+1/2}2^n}\sum_{m=0}^{n}\frac{m!(2\alpha)^{2m}}{(2m+1)!(n-m)!}\right].$$

**Proof.** We have

$$\Psi_Y(t) = \mathrm{E}[\exp\{\mathrm{i}t(\xi + W(\eta)^{1/2}X)\}] = \exp(\mathrm{i}\xi t)\mathrm{E}\{\exp(\mathrm{i}tW(\eta)^{1/2}X)\}$$
$$= \exp(\mathrm{i}\xi t)\left[\mathrm{E}\{\cos(tW(\eta)^{1/2}X)\} + \mathrm{i}\mathrm{E}\{\sin(tW(\eta)^{1/2}X)\}\right].$$

Then:

$$\mathrm{E}\{\cos(tW(\eta)^{1/2}X)\} = \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}\mathrm{E}(W(\eta)^n) \times \mathrm{E}(X^{2n}) \text{ by independence}$$
$$= \sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2n)!}c_{2n}\omega^{2n}\{1 \times 3 \times \cdots \times (2n-1)\}$$

after applying Lemma 2 and the moments of mixing distributions. Similarly, we use a Taylor series expansion of $\sin(\cdot)$ to finish the proof. □

When $\alpha = 0$, we have the characteristic function of scale mixtures of normal distributions.

**Corollary 3.** *Let $Y$ follow a scale mixture of normal distributions, i.e. $Y = \xi + W(\eta)^{1/2}X$, where $X$ follows the normal distribution, $X \sim N(0, \omega^2)$, and $\eta$ is a mixing variable with cdf $H(\eta)$ and a weight function $W(\eta)$ independent of $X$. Then the characteristic function of $Y$ is:*

$$\Psi_Y(t) = \exp(\mathrm{i}\xi t)\left[\sum_{n=0}^{\infty}\frac{(-1)^n(\omega t)^{2n}c_{2n}}{(2n)!}\{1 \times 3 \times \cdots \times (2n-1)\}\right]. \quad (21)$$

After some simple algebra, (21) equals the well-known characteristic function of the normal distribution when $W(\eta) = 1$.

**Corollary 4.** *Let $Y = \xi + X$, where $X$ has a normal distribution, $X \sim N(0, \omega^2)$. Then the characteristic function of $Y$ is:*

$$\Psi_Y(t) = \exp\left(\mathrm{i}\xi t - \omega^2 t^2/2\right).$$

Using the first order approximation by Taylor series expansions of $\sin(\cdot)$ and $\cos(\cdot)$, we have the following corollary.

**Corollary 5.** *Let $Y$ follow a scale mixture of skew-normal distributions, i.e. $Y = \xi + W(\eta)^{1/2}X$, where $X$ follows a skew-normal distribution, $X \sim \mathrm{SN}(0, \omega^2, \alpha)$, and $\eta$ is a mixing variable with cdf $H(\eta)$ and a weight function $W(\eta)$ independent of $X$. Then the first order approximate characteristic function of $Y$ is:*

$$\Psi_Y(t) \approx \exp(\mathrm{i}\xi t)\left(1 + \mathrm{i}tc_1\sqrt{\frac{2}{\pi}}\delta\omega\right).$$

In general, we can derive the characteristic function of scale mixtures of skew-normal distributions as mixtures of the characteristic function of skew-normal distributions.

**Theorem 4.** *Let $Y$ satisfy the stochastic equation (5). Then the characteristic function of $Y$ is:*

$$\Psi_Y(t) = \exp(\mathrm{i}\xi t)\int_0^{\infty}\Psi_Z\left(W(\eta)^{1/2}\omega t\right)\mathrm{d}H(\eta),$$

*where $\Psi_Z(t)$ is the characteristic function of $Z \sim \mathrm{SN}(\alpha)$.*

**Proof.** We have $Y|\eta \sim \mathrm{SN}(\xi, W(\eta)\omega^2, \alpha)$. Then:

$$\begin{aligned}\Psi_Y(t) &= \mathrm{E}\{\exp(\mathrm{i}tY)\} = \mathrm{E}_\eta[\mathrm{E}_Y\{\exp(\mathrm{i}tY)|\eta\}] \\ &= \int_0^\infty \int_{\mathbb{R}} \exp(\mathrm{i}ty) f(y|\eta)\mathrm{d}y\mathrm{d}H(\eta) \\ &= 2\exp(\mathrm{i}\xi t)\int_0^\infty \exp\left(-W(\eta)\omega^2 t^2/2\right)\Phi\left(\mathrm{i}W(\eta)^{1/2}\delta\omega t\right)\mathrm{d}H(\eta)\end{aligned}$$

by Theorem 1. The result follows directly by properties of the characteristic function. Note that $\Psi_Z(t) = 2\exp\left(-t^2/2\right)\Phi(\mathrm{i}\delta t)$. □

Using (13), the characteristic function of the skew-normal distribution, we have another version of the characteristic function of scale mixtures of skew-normal distributions.

**Theorem 5.** *Let Y satisfy the stochastic equation (5). Then the characteristic function of Y is:*

$$\Psi_Y(t) = \exp(\mathrm{i}\xi t)\left\{\int_0^\infty \Psi_U\left(W(\eta)^{1/2}\omega t\right)\mathrm{d}H(\eta) + \mathrm{i}\tau^*(\delta, \omega t)\right\}, \tag{22}$$

*where $\Psi_U(t)$ is the characteristic function of $U \sim N(0, 1)$ and $\tau^*(\cdot, \cdot)$ is defined in (20).*

**Proof.** The proof is similar to that of Theorem 4. The characteristic function of the standard normal distribution is $\Psi_U(t) = \exp\left(-t^2/2\right)$. □

A consequence of Theorem 5 is that the first part of (22) is the characteristic function of scale mixtures of normal distributions. So we can use these facts to derive the characteristic function of scale mixtures of skew-normal distributions. Lukacs [21] lists some basic characteristic functions. One special case is given in Theorem 2 for skew-$t$ distributions.

## 4. Characteristic functions: multivariate case

### 4.1. Skew-normal distributions

The characteristic function of a $k$-dimensional random vector $X$ is the function $\Psi_X : \mathbb{R}^k \to \mathbb{C}$ defined by $\Psi_X(t) = \mathrm{E}\{\exp(\mathrm{i}t^TX)\}$, for all $t \in \mathbb{R}^k$. The characteristic function of the multivariate skew-normal distribution is described in the next theorem.

**Theorem 6.** *Let $X \sim \mathrm{SN}_k(\xi, \Omega, \alpha)$. Then the characteristic function of X is:*

$$\begin{aligned}\Psi_X(t) &= 2\exp\left(\mathrm{i}t^T\xi - t^T\Omega t/2\right)\Phi(\mathrm{i}\delta^T\omega t) \\ &= \exp\left(\mathrm{i}t^T\xi - t^T\Omega t/2\right)\left\{1 + \mathrm{i}\tau(\delta^T wt)\right\},\end{aligned} \tag{23}$$

*where $\delta = \Omega_z\alpha/(1 + \alpha^T\Omega_z\alpha)^{1/2}$.*

**Proof.** Here we also provide three methods to prove this theorem.

*Method* 1: Since $M_X(s) = \mathrm{E}\{\exp(s^TX)\} = 2\exp\left(s^T\xi + s^T\Omega s/2\right)\Phi(\delta^T\omega s)$, $s = x + \mathrm{i}y, x, y \in \mathbb{R}^k$, is an analytic continuation of the multivariate skew-normal moment generating function, we have that $\Psi_X(t) = \mathrm{E}\{\exp(\mathrm{i}t^TX)\} = M_X(\mathrm{i}t)$. By Lemma 1, the rest of the proof is similar to the univariate case.

*Method* 2: Using the stochastic representation (10), we have:

$$\Psi_Z(t) = \mathrm{E}\{\exp(\mathrm{i}t^T\Delta 1_k|V_0|)\}\mathrm{E}[\exp\{\mathrm{i}t^T(I_k - \Delta^2)^{1/2}V_1\}]$$

by independence of $V_0$ and $V_1$. Then:

$$\mathrm{E}\{\exp(\mathrm{i}t^T\Delta 1_k|V_0|)\} = \exp\left(-t^T\Delta 1_k 1_k^T\Delta t/2\right)\{1 + \mathrm{i}\tau(t^T\Delta 1_k)\}$$

by the characteristic function of the half-normal distribution which was developed in Corollary 1. Since $V_1 \sim N_k(0, R)$,

$$\mathrm{E}[\exp\{\mathrm{i}t^T(I_k - \Delta^2)^{1/2}V_1\}] = \exp\left\{-t^T(I_k - \Delta^2)^{1/2}R(I_k - \Delta^2)^{1/2}t/2\right\}.$$

Hence, after some simple algebra and using the well-known relationship between the $(R, \Delta)$ and the $(\Omega_z, \alpha)$ set of parameters, we have that

$$\Psi_Z(t) = \exp\left(-t^T\Omega_z t/2\right)\left\{1 + \mathrm{i}\tau(t^T\delta)\right\}.$$

Therefore applying standard properties of the characteristic function to (8), we have proved the result.

*Method* 3: Treating $t^T\Delta 1_k$ as a constant, the proof is similar to the univariate case. □

The characteristic function of a finite mixture of multivariate skew-normal distributions is presented next.

**Corollary 6.** *Let Y follow a finite mixture of multivariate skew-normal distributions with pdf (12). Then the characteristic function of Y is:*

$$\begin{aligned}\Psi_Y(t) &= 2\exp(\mathrm{i}t^T\xi)\sum_{j=1}^{n} p_j \exp\left(-W(\eta_j)t^T\Omega t/2\right)\Phi(\mathrm{i}W(\eta_j)^{1/2}\delta^T\omega t)\\ &= \exp(\mathrm{i}t^T\xi)\sum_{j=1}^{n} p_j \exp\left(-W(\eta_j)t^T\Omega t/2\right)\left\{1+\mathrm{i}\tau(W(\eta_j)^{1/2}\delta^T\omega t)\right\}.\end{aligned}$$

**Proof.** The proof is similar to that of Theorem 6. □

One special case happens when $W(\eta) = 1/\eta$ and $H$ is a discrete measure on $\{\eta_1 = \gamma, \eta_2 = 1\}$ with probabilities $p, 1-p$, respectively. Hence by Corollary 6 the characteristic function of the multivariate skew-contaminated-normal distribution is:

$$\begin{aligned}\Psi_Y(t) &= 2\exp(\mathrm{i}t^T\xi)\left[p\exp(-\gamma^{-1}t^T\Omega t/2)\Phi(\mathrm{i}\gamma^{-1/2}\delta^T\omega t) + (1-p)\exp(-t^T\Omega t/2)\Phi(\mathrm{i}\delta^T\omega t)\right]\\ &= \exp(\mathrm{i}t^T\xi)\left[p\exp(-\gamma^{-1}t^T\Omega t/2)\left\{1+\mathrm{i}\tau\left(\gamma^{-1/2}\delta^T\omega t\right)\right\} + (1-p)\exp(-t^T\Omega t/2)\left\{1+\mathrm{i}\tau\left(\delta^T\omega t\right)\right\}\right].\end{aligned}$$

### 4.2. Skew-t distributions

The characteristic function of the multivariate skew-$t$ distribution is obtained using scale mixtures of multivariate skew-normal distributions and the Laplace–Stieltjes transform. The multivariate skew-$t$ distribution is related to the multivariate skew-normal distribution by the following stochastic equation:

$$Y = \xi + \eta^{1/2}X, \tag{24}$$

where $X$ has a multivariate skew-normal distribution, $X \sim \mathrm{SN}_k(0, \Omega, \alpha)$, and $\eta$ has an inverse-Gamma distribution, $\eta \sim \mathrm{IG}(\nu/2, \nu/2)$.

**Theorem 7.** *Let Y follow the multivariate skew-t distribution defined by (24). Then the characteristic function of Y is:*

$$\Psi_Y(t) = \exp(\mathrm{i}t^T\xi)\left[\Psi_{T_k}(\Omega^{1/2}t) + \mathrm{i}\tau^+(\delta, \omega t)\right],$$

*where*

$$\Psi_{T_k}(t) = \frac{\|\sqrt{\nu}t\|^{\nu/2}}{\Gamma(\nu/2)2^{\nu/2-1}}K_{\nu/2}(\|\sqrt{\nu}t\|), \quad \text{for } t \in \mathbb{R}^k,\ \nu > 0,$$

$$\tau^+(\delta, \omega t) = \int_0^\infty \exp(-\eta t^T\Omega t/2)\tau(\sqrt{\eta}\delta^T\omega t)\mathrm{d}H(\eta), \quad \text{for } \delta^T\omega t > 0, \tag{25}$$

*with* $\tau^+(\delta, -\omega t) = -\tau^+(\delta, \omega t)$.

**Proof.** The conditional distribution of $Y$ given $\eta$ follows a multivariate skew-normal distribution, i.e. $Y|\eta \sim \mathrm{SN}_k(\xi, \eta\Omega, \alpha)$. Then the characteristic function of $Y$ is

$$\begin{aligned}\Psi_Y(t) &= \int_0^\infty\int_{\mathbb{R}} \exp(\mathrm{i}t^Ty)f(y|\eta)\mathrm{d}y\mathrm{d}H(\eta) = \int_0^\infty \Psi_{Y|\eta}(t)\mathrm{d}H(\eta)\\ &= \exp(\mathrm{i}t^T\xi)\int_0^\infty \exp\left(-\eta t^T\Omega t/2\right)\left\{1+\mathrm{i}\tau(\sqrt{\eta}\delta^T\omega t)\right\}\mathrm{d}H(\eta)\\ &= \exp(\mathrm{i}t^T\xi)\left[L_\eta\left(t^T\Omega t/2\right) + \mathrm{i}\int_0^\infty \exp\left(-\eta t^T\Omega t/2\right)\tau(\sqrt{\eta}\delta^T\omega t)\mathrm{d}H(\eta)\right],\end{aligned}$$

where $L_\eta(\gamma)$ is the Laplace–Stieltjes transform when $\eta$ is a non-negative random variable. So $L_\eta\left(t^T\Omega t/2\right)$ becomes the characteristic function of the multivariate $t_k$ distribution with degrees of freedom $\nu$, $\Psi_{T_k}(\Omega^{1/2}t)$, after developing the characteristic function of the symmetric generalized hyperbolic distribution and then using some properties of the modified Bessel function of the third kind [17,18]. The remaining calculations are straightforward and the result follows. □

As a by-product, we have just derived the characteristic function of the multivariate $t$ distribution in a new way using the Laplace–Stieltjes transform. Hurst [17] only derived the univariate characteristic function of Student's $t$ distribution using the Laplace–Stieltjes transform. We found that his approach can also be extended to the multivariate situation. A particular case of the multivariate skew-$t$ distribution is the multivariate skew-Cauchy distribution, when $\nu = 1$. Also when $\nu \to \infty$, we obtain the multivariate skew-normal distribution as the limiting case. Hence, we have computed the characteristic functions of the multivariate skew-Cauchy and skew-normal distributions. For the characteristic function of the multivariate $t$ distribution, see [18].

### 4.3. Scale mixtures of multivariate skew-normal distributions

Similar to the univariate case, we derive the characteristic function of scale mixtures of multivariate skew-normal distributions as mixtures of the characteristic function of multivariate skew-normal distributions. One special case was given in Theorem 7 for the multivariate skew-$t$ distribution.

**Theorem 8.** *Let $Y$ satisfy the stochastic equation (11). Then the characteristic function of $Y$ is:*

$$\Psi_Y(t) = \exp(\mathrm{i}t^T\xi)\int_0^\infty \Psi_Z\left(W(\eta)^{1/2}\omega t\right)\mathrm{d}H(\eta),$$

*where $\Psi_Z(t)$ is the characteristic function of $Z \sim \mathrm{SN}_k(\Omega_z, \alpha)$.*

**Proof.** We have $Y|\eta \sim \mathrm{SN}_k(\xi, W(\eta)\Omega, \alpha)$. Then:

$$\begin{aligned}\Psi_Y(t) &= \mathrm{E}\{\exp(\mathrm{i}t^TY)\} = \mathrm{E}_\eta[\mathrm{E}_Y\{\exp(\mathrm{i}t^TY)|\eta\}] \\ &= \int_0^\infty\int_{\mathbb{R}^k}\exp(\mathrm{i}t^Ty)f(y|\eta)\mathrm{d}y\mathrm{d}H(\eta) \\ &= 2\exp(\mathrm{i}t^T\xi)\int_0^\infty \exp\left(-W(\eta)t^T\Omega t/2\right)\Phi\left(\mathrm{i}W(\eta)^{1/2}\delta^T\omega t\right)\mathrm{d}H(\eta)\end{aligned}$$

by Theorem 6. The result follows directly by properties of characteristic functions. Note that $\Psi_Z(t) = 2\exp\left(-t^T\Omega_z t/2\right)\Phi(\mathrm{i}\delta^T t)$. □

Using (23), the characteristic function of the multivariate skew-normal distribution, we have a more informative characteristic function of the scale mixtures of multivariate skew-normal distributions.

**Theorem 9.** *Let $Y$ satisfy the stochastic equation (11). Then the characteristic function of $Y$ is:*

$$\Psi_Y(t) = \exp(\mathrm{i}t^T\xi)\left\{\int_0^\infty \Psi_U\left(W(\eta)^{1/2}\Omega^{1/2}t\right)\mathrm{d}H(\eta) + \mathrm{i}\tau^+(\delta, \omega t)\right\},$$

*where $\Psi_U(t)$ is the characteristic function of $U \sim N_k(0, I_k)$ and $\tau^+(\cdot,\cdot)$ is defined in (25).*

**Proof.** The proof is similar to that of Theorem 8. The characteristic function of the standard multivariate normal distribution is $\Psi_U(t) = \exp\left(-t^Tt/2\right)$. □

A consequence of Theorem 9 is that the first part of the characteristic function of $Y$ is the characteristic function of scale mixtures of multivariate normal distributions. So we can use these facts to derive the characteristic function of scale mixtures of multivariate skew-normal distributions. One special case is given in Theorem 7 for multivariate skew-$t$ distributions.

## Acknowledgments

The authors thank Reinaldo Arellano-Valle, Cornelis Potgieter, and an anonymous referee for valuable comments. Genton's research was supported in part by NSF grant DMS-1007504 and by Award No. KUS-C1-016-04, made by King Abdullah University of Science and Technology (KAUST).

## References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, Dover Publications, Inc., New York, 1972
[2] R.B. Arellano-Valle, A. Azzalini, On the unification of families of skew-normal distributions, Scandinavian Journal of Statistics 33 (2006) 561–574.
[3] R.B. Arellano-Valle, M.D. Branco, M.G. Genton, A unified view on skewed distributions arising from selections, The Canadian Journal of Statistics 34 (2006) 581–601.
[4] R.B. Arellano-Valle, M.G. Genton, On fundamental skew distributions, Journal of Multivariate Analysis 96 (2005) 93–116.
[5] R.B. Arellano-Valle, M.G. Genton, An invariance property of quadratic forms in random vectors with a selection distribution, with application to sample variogram and covariogram estimators, Annals of the Institute of Statistical Mathematics 62 (2010) 363–381.
[6] A. Azzalini, A class of distributions which includes the normal ones, Scandinavian Journal of Statistics 12 (1985) 171–178.
[7] A. Azzalini, The skew-normal distribution and related multivariate families (with discussion), Scandinavian Journal of Statistics 32 (2005) 159–188. C/R 189-200.
[8] A. Azzalini, A. Capitanio, Statistical applications of the multivariate skew normal distribution, Journal of the Royal Statistical Society, Series B 61 (1999) 579–602.
[9] A. Azzalini, A. Capitanio, Distributions generated by perturbation of symmetry with emphasis on a multivariate skew $t$ distribution, Journal of the Royal Statistical Society, Series B 65 (2003) 367–389.
[10] A. Azzalini, A. Dalla Valle, The multivariate skew-normal distribution, Biometrika 83 (1996) 715–726.
[11] A. Azzalini, M.G. Genton, Robust likelihood methods based on the skew-$t$ and related distributions, International Statistical Review 76 (2008) 106–129.
[12] T.M. Bisgaard, Z. Sasvári, Characteristic Functions and Moment Sequences, Nova Science Publishers, Inc., New York, 2000
[13] M.D. Branco, D.K. Dey, A general class of multivariate skew-elliptical distributions, Journal of Multivariate Analysis 79 (2001) 99–113.

[14] D.J. Buckle, Bayesian inference for stable distributions, Journal of the American Statistical Association 90 (1995) 605–613.
[15] M.G. Genton, Skew-Elliptical Distributions and their Applications: A Journey Beyond Normality, Edited Volume, Chapman & Hall, CRC, Boca Raton, FL, 2004.
[16] N. Henze, A probabilistic representation of the skew-normal distribution, Scandinavian Journal of Statistics 13 (1986) 271–275.
[17] S. Hurst, The characteristic function of the Student $t$-distribution, in: Financial Mathematics Research Report 006-95, Australian National University, Canberra ACT 0200, Australia, 1995.
[18] S. Kotz, S. Nadarajah, Multivariate t Distributions and their Applications, Cambridge University Press, Cambridge, 2004.
[19] T.J. Kozubowski, K. Podgórski, Skew Laplace distributions. I. Their origins and inter-relations, Mathematical Scientist 33 (2008) 22–34.
[20] V.H. Lachos, D.K. Dey, V.G. Cancho, Robust linear mixed models with skew-normal independent distributions from a Bayesian perspective, Journal of Statistical Planning and Inference 139 (2009) 4098–4110.
[21] E. Lukacs, Characteristic Functions, Griffin, London, 1970.
[22] E. Lukacs, Developments in Characteristic Function Theory, Macmillan Publishing Co., Inc., New York, 1983
[23] P.R. Nelson, An approximation for the complex normal probability integral, BIT Numerical Mathematics 22 (1982) 94–100.
[24] A. Pewsey, The wrapped skew-normal distribution on the circle, Communications in Statistics—Theory and Methods 29 (2000) 2459–2472.
[25] G. Samorodnitsky, M.S. Taqqu, Stable Non-Gaussian Random Processes, Chapman and Hall, New York, 1994.
[26] G.P. Steck, B. Owen, A note on the equicorrelated multivariate normal distribution, Biometrika 49 (1962) 269–271.
[27] J. Wang, M.G. Genton, The multivariate skew-slash distribution, Journal of Statistical Planning and Inference 136 (2006) 209–220.