Nonparametric Identification of Copula Structures

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We propose a unified framework for testing a variety of assumptions commonly made about the structure of copulas, including symmetry, radial symmetry, joint symmetry, associativity and Archimedeanity, and max-stability. Our test is nonparametric and based on the asymptotic distribution of the empirical copula process. We perform simulation experiments to evaluate our test and conclude that our method is reliable and powerful for assessing common assumptions on the structure of copulas, particularly when the sample size is moderately large. We illustrate our testing approach on two datasets.

KEY WORDS: Archimedeanity; Associativity; Asymptotic normality; Max-stability; Symmetry; Test.

1. INTRODUCTION

Modeling multivariate data with copulas has been used in many applications in recent years, especially in actuarial science, finance, hydrology, and survival analysis; see Genest and Favre (2007) for a review and references therein. In particular, the use of copulas in financial areas such as asset pricing and credit risk management has soared; see, for example, the books by Cherubini, Luciano, and Vecchiato (2004) and by McNeil, Frey, and Embrechts (2005). The apparent success of copulas is due to their ability to model the dependence structure of a random vector separately from its marginal behavior. Indeed, because of the representation theorem provided by Sklar (1959), every multivariate cumulative distribution function, \( F \), of a continuous random vector, \( X = (X_1, \ldots, X_d)^T \) on \( \mathbb{R}^d \), can be written as

\[
F(x_1, \ldots, x_d) = \Pr(X_1 \leq x_1, \ldots, X_d \leq x_d) = C(F_1(x_1), \ldots, F_d(x_d)),
\]

where \( F_j(x_j) = \Pr(X_j \leq x_j), \) for \( j \in \{1, \ldots, d\} \), are the continuous marginal distributions, and \( C : [0,1]^d \rightarrow [0,1] \) is the unique copula. The latter contains all the information about the dependence structure of the random vector, \( X \), and can be obtained from

\[
C(u_1, \ldots, u_d) = \Pr(U_1 \leq u_1, \ldots, U_d \leq u_d) = F^{-1}(u_1, \ldots, F_d^{-1}(u_d)),
\]

where \( F_j^{-1}(u_j) = \inf \{ x | F_j(x) \geq u_j \} \), for \( j \in \{1, \ldots, d\} \), is the quantile function of \( F_j \) for \( j = 1, \ldots, d \), and \( U = (U_1, \ldots, U_d)^T \) with \( U_j = F_j(X_j) \), for \( j = 1, \ldots, d \). From (2), the copula \( C \) is itself a cumulative distribution function, of \( U \), on the \( d \)-dimensional unit hypercube with uniform marginal distributions.

From a practical point of view, the representation in Equation (1) allows the dependence structure to be modeled given the marginal distributions by choosing one of the many parametric copula models available in the literature; see, for example, the books by Joe (1997), Nelsen (2006), and Panjer (2006) for the most common parametric families of copulas. Mikosch (2006) then asked the following natural question: How does one choose a copula? In those days, copula models were often chosen by subject-matter specialists for mathematical convenience rather than usefulness for the data at hand. The main purpose of this article is to change this procedure by proposing a new and nonparametric method that allows the data to choose the copula model.

One model selection approach that has recently received sustained attention is to consider goodness-of-fit tests for copulas. The procedure consists of testing that the unknown copula, \( C \), belongs to a parametric family, \( C_0 = \{ C_\theta \}_{\theta \in \Theta} \subset \mathbb{R}^p \), of copulas, that is, \( H_0 : C \in C_0 \) versus \( H_1 : C \notin C_0 \). Tests aimed at discriminating between \( H_0 \) and \( H_1 \) have been proposed; see the reviews by Genest, Rémillard, and Beaudoin (2009), Berg (2009), and references therein. These methods consider the univariate marginal distributions as infinite-dimensional nuisance parameters, and typically in that context, the observations are replaced by the maximally invariant statistics, that is, the ranks. Other attempts for choosing parametric families of copulas can be found in the literature. For example, Craiu and Craiu (2008) investigated choices based on various distances between a nonparametric density estimate and the density implied by parametric copula models fitted by maximum likelihood. Michiels and De Schepper (2008) introduced test space models for copulas, that is, comparable families of copulas with respect to their dependence range.

Our approach is different from the previous ones because we aim at testing the structure of copulas. Our test can be viewed as complementary to the above goodness-of-fit methods because it will suggest sensible classes, \( C_0 \), of parametric families of copulas. Very recently, attempts to specify copula structures emerged in several articles. Jaworski (2010) proposed a simple test for the associativity structure based on the asymptotic distribution of the pointwise copula estimator. The disadvantage of this test is that it only tests the associativity at a particular point rather than for the whole copula process; see the discussion in Bücher, Dette, and Volgushev (2012). Bücher, Dette, and Volgushev (2012) derived Cramér-von Mises and Kolmogorov–Smirnov type test statistics for assessing the characteristics of associativity, based on which they further formed a test statistic for Archimedeanity. Bücher, Dette, and Volgushev (2011) developed a test for extreme-value dependence through...
the unique property of the minimum weighted \( L^2 \)-distance of extreme-value copulas. A test for bivariate symmetry of copulas based on Cramér-von Mises and Kolmogorov–Smirnov functionals of the rank-based empirical copula process was introduced by Genest, Nešlehová, and Quesy (2012), but it does not provide more specific guidance for model choice (see also Kojadinovic and Yan 2012). Other specific tests of structures have been reviewed by Genest and Nešlehová (2012, 2013; see also references therein). In contrast, we propose a unified framework for testing a variety of assumptions commonly made for the structure of copulas, including symmetry, radial symmetry, joint symmetry, associativity and Archimedeanity, and max-stability. Our test is nonparametric and based on the asymptotic distribution of the empirical copula process.

The article is organized as follows. In Section 2, we describe various structures of copulas that are of interest to applications, such as symmetries, Archimedeanity, max-stability, and others. In particular, we provide a novel schematic representation of those structures and interrelations for various popular copula models in Figure 1. In Section 3, we introduce our nonparametric testing procedures based on the asymptotic distribution of the empirical copula. We report the results of Monte Carlo simulations in Section 4 where we investigate the empirical sizes and powers of our tests, as well as the choice of testing points. In Section 5, we present illustrations of our methodology on two applications. The article ends with a discussion in Section 6.

2. COPULA STRUCTURES

2.1 Copulas With Symmetry Structures

For the sake of simplicity, we follow Nelsen (1993) and focus on the case \( d = 2 \), although the results can be extended to \( d > 2 \). There are various concepts of bivariate symmetry for copulas. The simplest notion is to say that a copula, \( C \), is symmetric if

\[
C(u_1, u_2) = C(u_2, u_1) = 0,
\]

for all \((u_1, u_2) \in [0, 1]^2\). Many copulas satisfy this structure of symmetry. Moreover, two random variables are exchangeable if and only if their marginal distributions are the same and their copula is symmetric. Another concept of symmetry is to say that a copula, \( C \), is radially symmetric if

\[
C(u_1, u_2) = C(1 - u_1, 1 - u_2) + 1 - u_1 - u_2 = 0,
\]

for all \((u_1, u_2) \in [0, 1]^2\). There exist copulas that are symmetric but not radially symmetric and, conversely, copulas that are radially symmetric but not symmetric. In particular, exchangeability does not imply radial symmetry and vice versa. Finally, we say that a copula, \( C \), is jointly symmetric if

\[
C(u_1, u_2) + C(1 - u_1, u_2) - u_1 = 0
\]

and

\[
C(u_1, u_2) + C(1 - u_1, u_2) - u_2 = 0,
\]

for all \((u_1, u_2) \in [0, 1]^2\). Jointly symmetric copulas are necessarily radially symmetric, but they could be symmetric or not. Figure 1 depicts these notions of symmetry structures and their interrelations. More discussions on bivariate symmetry concepts can be found in Nelsen (1993).

2.2 Copulas With Archimedean Structures

Archimedean copulas are an important class of copulas. Their popularity arises from their mathematical convenience and attractive properties. Specifically, a copula is Archimedean if it takes the form

\[
C\phi(u_1, \ldots, u_d) = \phi^{1-d}(\phi(u_1) + \cdots + \phi(u_d)),
\]

for all \((u_1, \ldots, u_d) \in [0, 1]^d\) and for some convex, continuous and strictly decreasing function \( \phi : [0, 1] \rightarrow [0, \infty] \) such that \( \phi(1) = 0 \). The function, \( \phi \), is called the generator of the copula, for which the (pseudo) inverse generator, \( \phi^{1-d}(t) \), must be a completely monotonic function. This general class encompasses many well-known copulas; see for instance Genest and MacKay (1986a,b), Joe (1997), and Nelsen (2006).

Archimedean copulas are symmetric and associative, that is, for \( d = 2 \), \( C(u_1, u_2) = C(u_2, u_1) \) and \( C(C(u_1, u_2), u_3) = C(u_1, C(u_2, u_3)) \) for all \((u_1, u_2, u_3) \in [0, 1]^3\). Indeed, an Archimedean copula is characterized by an associative copula with \( \delta(c)(u) < u \) for all \( u \in (0, 1) \), where \( \delta(c)(u) = C(u, u) \) is the diagonal section of a copula and \( \delta(c)(u) = \phi^{1-d}(2\phi(u)) \) for an Archimedean copula, \( C \), with generator \( \phi \), see Ling (1965).

2.3 Copulas With Max-Stable Structures

A copula, \( C \), is said to be max-stable, or equivalently to be an extreme-value copula, if

\[
C(u_1, \ldots, u_d) = C'(u_1^{1/r}, \ldots, u_d^{1/r}) = 0,
\]

for all \((u_1, \ldots, u_d) \in [0, 1]^d\) and for any \( r > 0 \). Such copulas appear naturally as the copula \( C(u_1, \ldots, u_d) = \lim_{n \rightarrow \infty} \hat{C}_n(u_1^{1/n}, \ldots, u_d^{1/n}) \) associated with componentwise maxima of \( n \) independent and identically distributed random vectors with copula \( \hat{C} \).

A related structure is that a copula, \( C \), is homogeneous of degree \( k \in \mathbb{R} \), that is,

\[
C(\lambda u_1, \ldots, \lambda u_d) = \lambda^k C(u_1, \ldots, u_d) = 0,
\]

for all \((u_1, \ldots, u_d) \in [0, 1]^d\) and for any \( 0 \leq \lambda \leq 1 \). An interesting result is that if \( C \) is a homogeneous copula of degree \( k \), then necessarily \( 1 \leq k \leq d \) and \( C \) has to belong to the Cuadras–Augé family with parameter \( \theta = d - k \), which are symmetric and max-stable. The proof of this result for \( d = 2 \) can be found in Nelsen (2006, p. 102) and its extension to \( d > 2 \) is straightforward.

2.4 Relations Between Copula Structures

Copulas satisfy a version of the Fréchet–Hoeffding bounds inequality. Specifically, when \( d = 2 \), for every copula \( C \) and every \((u_1, u_2) \in [0, 1]^2\):

\[
W(u_1, u_2) \leq C(u_1, u_2) \leq M(u_1, u_2),
\]

where \( W(u_1, u_2) = \max(u_1 + u_2 - 1, 0) \) and \( M(u_1, u_2) = \min(u_1, u_2) \) are themselves copulas, and both are symmetric and radially symmetric. In addition, \( W \) is Archimedean and \( M \) is max-stable. Although these two copulas are too simplistic for real applications, they play a major theoretical role as bounds. Another simple yet important copula is the product copula \( \Pi(u_1, u_2) = u_1 u_2 \) that corresponds to the independence case. It satisfies all aforementioned notions of symmetries and is both Archimedean and max-stable.

The elliptical copulas (Fang, Fang, and Kotz 2002) are those associated with an elliptically contoured distribution, \( F \), in (2) (Fang, Kotz, and Ng 1990). They are symmetric and radially
symmetric. Two important members of this family are the Gaussian copula and the Student’s $t$ copula (Demarta and McNeil 2005). Skew-elliptical copulas are those associated with skew-elliptical distributions, see the book edited by Genton (2004) for an overview. As their name suggests, they are asymmetric copulas. In particular, skew-$t$ copulas have been studied by Kollo and Pettte (2010). Other asymmetric copulas have been constructed by Liebscher (2008).

The Farlie–Gumbel–Morgenstern copula (Farlie 1960; Gumbel 1958; Morgenstern 1956) and the Plackett copula (Plackett 1965) are both symmetric and radially symmetric. Nelsen (1993) studied examples of copulas with different symmetry structures. In his figure 1, copulas #2 and #8 are both symmetric and radially symmetric, and copula #1 is in addition also jointly symmetric. Copulas #5 and #11 are only radially symmetric, whereas copula #4 is both radially and jointly symmetric. These copulas are presented in our Figure 1 for $d = 2$.

Some of the most important members of the family of Archimedean copulas are the Clayton copula (Clayton 1978), the Frank copula (Frank 1979), the Gumbel–Hougaard copula (Gumbel 1960), the Ali-Mikhail-Haq copula (Ali, Mikhail, and Haq 1978), and the Gumbel–Barnet copula (Gumbel 1960; Barnett 1980). They are all symmetric by the nature of the Archimedean form. The Frank copula is also radially symmetric. The Gumbel–Hougaard copula satisfies the max-stable condition as well. In fact, it is the only copula that is both Archimedean and max-stable (Nelsen 2006, p. 143).

Several copulas are symmetric and max-stable, including the Cuadras–Augé family (Cuadras and Augé 1981) and the symmetric Galambos copula (Galambos 1975), although there is also an asymmetric version. Two copulas that are asymmetric and max-stable are the Marshall–Olkin family of copulas (Marshall and Olkin 1967), also known as the generalized Cuadras–Augé family, and the Tawn copula (Tawn 1988) of which the symmetric Gumbel–Hougaard copula is a special case.

3. TESTING PROCEDURES

3.1 Asymptotics

Let $u = (u_1, \ldots, u_d)^T$ and let $\Lambda$ be a set of user-chosen points in the unit hypercube $[0, 1]^d$ of cardinality $c$. Typically, $\Lambda$ is a regular discretization of the interior of $[0, 1]^d$, and the
appropriate choice of \(c\) will be investigated in Section 4.2. Define \(G = \{C(u) : u \in \Lambda\}\) to be the length, \(c\), column vector of the copula, \(C\), evaluated at the points in \(\Lambda\).

Given the marginals, \(U\), we can choose a moment estimator for \(C(u)\) such that
\[
\hat{C}_n(u) = \frac{1}{n} \sum_{i=1}^{n} \prod_{k=1}^{d} I(U_{ki} \leq u_k),
\]
where \(I(\cdot)\) denotes the indicator function. Let \(\hat{G}_n = \{\hat{C}_n(u) : u \in \Lambda\}\). It has been shown (e.g., Tsukahara 2005; Rémillard and Scaillet 2009; Genest and Segers 2010) that
\[
\sqrt{n}(\hat{G}_n(u) - C(u)) \xrightarrow{d} U(u),
\]
as \(n \to \infty\), where \(U(u)\) is a \(d\)-dimensional pinned C-Brownian sheet; that is, it is a centered Gaussian random field with \(\text{cov}(U^C(u), U^C(u')) = C(u, u') - C(u)C(u')\), where \(u_i\) and \(u_j\) denotes the componentwise minimum. Then, we have
\[
\sqrt{n}(\hat{G}_n(u) - C(u)) \xrightarrow{d} N_c(0, \Sigma),
\]
as \(n \to \infty\), where
\[
\Sigma_{ij} = \text{cov}(U^C(u_i), U^C(u_j)) = C(u_i, u_j) - C(u_i)C(u_j). \quad (8)
\]
Given a known copula, we can directly calculate \(\Sigma_{ij}\). If the form of the copula is unknown, we choose to use the plug-in estimator by replacing \(C(u)\) with its moment estimator \(\hat{G}_n(u)\).

It is often the case that the marginals, \(U\), are unknown and we only observe the data points \(X = (X_1, \ldots, X_d)^T\). For each \(X_i = (X_{i1}, \ldots, X_{id})^T, i = 1, \ldots, n\), we can estimate its corresponding \(U_i = (U_{i1}, \ldots, U_{id})^T\) by the moment estimators \(\hat{U}_i = (\hat{U}_{i1}, \ldots, \hat{U}_{id})^T\), where \(\hat{U}_{ki} = \frac{1}{n} \sum_{j=1}^{n} I(X_{kj} \leq X_{ki})\) for \(k = 1, \ldots, d\). Then, we can estimate \(C(u)\) again by the moment estimator but based on \(\hat{U}_i, i = 1, \ldots, n\). Let
\[
\hat{D}_n(u) = \frac{1}{n} \sum_{i=1}^{n} \prod_{k=1}^{d} I(\hat{U}_{ki} \leq u_k).
\]
It has been shown (Tsukahara 2005; Rémillard and Scaillet 2009; Genest and Segers 2010) that when \(C(u)\) is differentiable with continuous partial derivatives \(\frac{\partial C(u)}{\partial u_k}\) for \(k = 1, \ldots, d\),
\[
\sqrt{n}(\hat{D}_n(u) - C(u)) \xrightarrow{d} U(u) - \sum_{k=1}^{d} \frac{\partial C(u)}{\partial u_k} U^C(1, u_k, 1),
\]
as \(n \to \infty\), where \(1\) is a vector of ones with the proper length depending on the current context. Define \(\hat{H}_n = \{\hat{D}_n(u) : u \in \Lambda\}\) and \(H = G\). Then, it can be derived that
\[
\sqrt{n}(\hat{H}_n - H) \xrightarrow{d} N_c(0, \Pi),
\]
as \(n \to \infty\), where
\[
\Pi_{ij} = C(u_i \wedge u_j) - C(u_i)C(u_j)
\]
\[+ \sum_{m=1}^{d} \sum_{n=1}^{d} \frac{\partial C(u_i)}{\partial u_{mi}} \frac{\partial C(u_j)}{\partial u_{nj}} [C(1, u_{mi}, 1) \wedge (1, u_{nj}, 1)]
\]
\[- u_{mi}u_{nj}]. \quad (9)
\]
We again choose the plug-in method for estimating \(\Pi\). The unknown \(C(u)\) can be estimated by \(\hat{D}_n(u)\), and we estimate the partial derivatives following the technique in Rémillard and Scaillet (2009). Specifically, let \(e_k\) be a length \(d\) vector of zeroes but with the \(k\)th element replaced by 1, \(\frac{\partial C(u)/\partial u_k}\) can be consistently estimated by \(\{\hat{D}_n(u + h_{1}e_k) - \hat{D}_n(u - h_{1}e_k)\}/(2h_{1})\). We choose \(h_{1} = n^{-1/4}\) following Bücher, Dette, and Volgushev (2012). Genest and Segers (2010) had shown that \(\Pi \preceq \Sigma\) componentwise under some moderate conditions of positive dependence.

To assess the Archimedean property for \(d = 2\), we have to estimate \(C(1, u_2, u_3)\) and \(C(u_1, 1, u_3)\) by \(\hat{C}_n(\hat{G}_n(u_2, u_3))\) or \(\hat{G}_n(\hat{C}_n(u_1, u_2, u_3))\) or \(\hat{D}_n(\hat{D}_n(u_1, u_2, u_3))\) or \(\hat{D}_n(\hat{D}_n(u_1, u_3), u_2)\) and \(\hat{D}_n(u_1, \hat{D}_n(u_2, u_3))\), respectively. Thus, we also investigate the asymptotics of those estimators. We denote \(\hat{G}_n = \{\hat{C}_n(\hat{G}_n(u_2, u_3)) : (u_1, u_2, u_3)^T \in \Lambda\}\) for known marginals, and \(\hat{H}_n = \{\hat{D}_n(\hat{D}_n(u_1, u_2, u_3)) : (u_1, u_2, u_3)^T \in \Lambda\}\) for unknown marginals. Let \(\hat{G}_0 = \{C(1, u_2, u_3) : (u_2, u_3)^T \in \Lambda\}\) and \(\hat{H}_0 = \{\hat{H}_n : (u_1, u_2, u_3)^T \in \Lambda\}\). Correspondingly, the elements in \(\hat{G}_0\) and \(\hat{H}_0\) will have \(C(1, u_2, u_3)\). We prove the following result for \(d = 2\) in the Appendix.

**Theorem 3.1.** If \(\frac{\partial C(u_1, u_2)}{\partial u_1} + \frac{\partial C(u_1, u_2)}{\partial u_2}\) exist and are continuous at all \((u_1, u_2) \in [0, 1]^2\), then for any \(\Lambda\) that is a set of \((u, v, w) \in [0, 1]^3\), we have
\[
\sqrt{n}(\hat{G}_n - G_0) \xrightarrow{d} N_c(0, \Phi),
\]
and
\[
\sqrt{n}(\hat{H}_n - H_0) \xrightarrow{d} N_c(0, \Psi),
\]
as \(n \to \infty\), for covariance matrices \(\Phi\) and \(\Psi\). The dimension \(c\) of the multinormal distribution is determined by the cardinality of \(\Lambda\) consisting of triplets.

Under unknown margins, Genest and Segers (2009) showed the consistency and asymptotic normality of rank-based estimators of the Pickands dependence function, which can be expressed as appropriate functionals of the empirical copula. Segers (2012) further demonstrated the weak convergence of the empirical copula process without requiring the existence or continuity of the partial derivatives on certain boundaries. The result in Theorem 3.1 is closely related to Genest and Segers (2009) and Segers (2012), but here we consider the asymptotics of empirical copulas at again empirical copulas rather than at constant probabilities. These results are also related to Bücher, Dette, and Volgushev (2012) who considered a particular empirical copula process based on the characteristics of associativity. In Theorem 3.1, the explicit forms of the two matrices, \(\Phi\) and \(\Psi\), are intricate as discussed in the Appendix. In practice, we therefore estimate them via bootstrap techniques. We can use a standard bootstrap procedure to estimate \(\Phi\), but when marginals are unknown, we resort to the multiplier bootstrap provided in Bücher, Dette, and Volgushev (2012) to estimate \(\Psi\).
3.2 Test Statistics

We now describe the statistics to test the various copula structures described in Section 2. For simplicity of exposition, we describe the bivariate setting of \(d = 2\). The symmetry structure (3) and the max-stability structure (7) of copulas can be rewritten in vector form:

\[
\begin{pmatrix} 1, -1 \end{pmatrix} \begin{pmatrix} C(u_1, u_2) \\ C(u_2, u_1) \end{pmatrix} = 0
\]

and

\[
(1, -1) \begin{pmatrix} C(u_1, u_2) \\ C^r(1^{1/r}, u_2^{1/r}) \end{pmatrix} = (1, -1) f \begin{pmatrix} C(u_1, u_2) \\ C(1^{1/r}, u_2^{1/r}) \end{pmatrix} = 0,
\]

for a function, \(f\). The radial and joint symmetry structures (4) and (5), respectively, only require obvious modifications of the above representation. For example, radial symmetry uses

\[
(1, -1) \begin{pmatrix} C(u_1, u_2) + 1 - u_1 - u_2 \\ C(1 - u_1, 1 - u_2) \end{pmatrix} = 0.
\]

Since the equality has to hold for any \((u_1, u_2) \in [0, 1]^2\) if the copula possesses the corresponding particular property, we extend the equalities to \(Af(G) = 0\) for a contrast matrix \(A\). Indeed, many properties can be written into such forms. For those properties, we can perform the hypothesis test by the following test statistic

\[
TS1 = n[Af(\hat{G}_n)]^T(AB^T \Sigma BA)^{-1}[Af(\hat{G}_n)],
\]

if the marginals, \(U\), are known. The contrast matrix \(A\) has the row rank \(q\) and the function \(f = (f_1, \ldots, f_s)^T\). The matrix \(B\) is defined as \(B_{ij} = \delta f_j / \delta G_i, i = 1, \ldots, c, j = 1, \ldots, s\). When the marginals, \(U\), need to be estimated from \(X\), we use the test statistic

\[
TS2 = n[Af(\hat{H}_n)]^T(AB^T \Pi BA)^{-1}[Af(\hat{H}_n)].
\]

Due to the asymptotic normality of \(\hat{G}_n\) and \(\hat{H}_n\), \(TS1 \stackrel{d}{\rightarrow} \chi^2_q\) and \(TS2 \stackrel{d}{\rightarrow} \chi^2_{2q}\) asymptotically. The choice of \(A\) and \(f\) depends on the individual tests, and both \(\Sigma\) and \(\Pi\) are estimated by the plug-in method.

If the hypothesis of symmetry structure is rejected, then the copula cannot be Archimedean. If it is not rejected, then the identification of the Archimedean copula structure (6) requires two tests as mentioned in Section 2.2: (i) Associativity and (ii) \(\delta_\xi(u) < u\).

Associativity requires \(C(C(u_1, u_2), u_3) = C(u_1, C(u_2, u_3))\) for all \((u_1, u_2, u_3) \in [0, 1]^3\), which can be rewritten as

\[
(1, -1) \begin{pmatrix} C(C(u_1, u_2), u_3) \\ C(u_1, C(u_2, u_3)) \end{pmatrix} = 0.
\]

Then,

\[
TS3 = n[\hat{A}(\hat{G}_n)]^T(\hat{A} \Phi \hat{A}^T)^{-1}[\hat{A}(\hat{G}_n)]
\]

can be used for testing the associativity when the marginals are known. For unknown marginals, we have the following test:

\[
TS4 = n[\hat{A}(\hat{H}_n)]^T(\hat{A} \Psi \hat{A}^T)^{-1}[\hat{A}(\hat{H}_n)].
\]

The asymptotic covariance matrices, \(\Phi\) and \(\Psi\), are defined in the Appendix. However, for practical purposes, in the associativity test, we estimate \(\Phi\) and \(\Psi\) using bootstrap resampling techniques due to their intricate form as discussed in the previous section.

Bücher, Dette, and Volgushev (2012) proposed a function, \(A_n(\hat{D}_n) = \max\{\eta(1 - \eta) : \hat{D}_n(\eta, \eta) = \eta\}\) for \(0 < \eta < 1\), to examine \(\delta_\xi(u) < u\) versus \(\delta_\xi(u) = u\). Under mild conditions, \(A_n(\hat{D}_n) \sim \eta(1 - \eta) + 0\_\eta(1 - \eta)\) if there exists an \(\eta\) such that \(C(\eta, \eta) = \eta\). They integrated \(A_n(\hat{D}_n)\) with their test for associativity to form the test statistic for assessing Archimedeanity. We can naturally borrow the same idea to combine \(A_n(\hat{D}_n)\) with our test for associativity into the following test statistic for Archimedeanity assessment:

\[
TS5 = TS4 + k_n \psi(A_n(\hat{D}_n)),
\]

where \(k_n \sim n^\alpha, \alpha \in (0, 1/2)\) is a user-chosen constant and \(\psi(\cdot)\) is an increasing function with \(\psi(0) = 0\).

The matrix \(A\) in all test statistics needs to be of full row rank to avoid the singularity problem. This is equivalent to requiring that all the contrasts formed by \(A\) be linearly independent. Particular attention is necessary for both the radial symmetry and joint symmetry tests because their corresponding contrasts formed over a set of testing points can possibly overlap. For example, the following contrasts for testing joint symmetry are linearly dependent contrasts because if any three of them hold, then the remaining one holds too:

\[
C(u_1, u_2) + C(u_1, 1 - u_2) - u_1 = 0;
\]

\[
C(u_1, 1 - u_2) + C(1 - u_1, 1 - u_2) - 1 + u_2 = 0;
\]

\[
C(u_1, u_2) + C(1 - u_1, u_2) - u_2 = 0;
\]

\[
C(1 - u_1, u_2) + C(1 - u_1, 1 - u_2) - 1 + u_1 = 0.
\]

There are virtually only three free contrasts, and thus the matrix \(A\) needs to filter out one of them to maintain full row rank.

4. MONTE CARLO SIMULATIONS

We conduct simulations to evaluate our testing procedures for various structures of copulas and meanwhile compare our method to several other tests that have been developed for assessing particular structures. Since the marginals are rarely known, we conduct all the simulations with unknown marginals. Specifically, we examine the tests for symmetry, radial symmetry, joint symmetry, associativity, and max-stability. All the simulations have 1000 replications, and the nominal level of all tests is 5% unless otherwise specified. We choose \(r = 2\) in testing the max-stable structure, and we choose 500 bootstrap samples in the associativity test. We compare our symmetry test to Genest, Nešlehová, and Quessy (2012), the associativity test to Bücher, Dette, and Volgushev (2012), and the max-stability test to Bücher, Dette, and Volgushev (2011). In these comparisons, we follow their choice of copula models and simulation settings.

4.1 Testing Results

Table 1 reports the sizes and powers of the test of symmetry. Following Genest, Nešlehová, and Quessy (2012), hereinafter GNJQ, three copula models are made asymmetric by Khoudraj’s device (Khoudraj 1995). Specifically, an asymmetric version of a copula, \(C(u_1, u_2)\), is defined at all \((u_1, u_2) \in [0, 1]^2\) by

\[
K_\delta(u_1, u_2) = u_1^{\delta} C(u_1^{1-\delta}, u_2).
\]
for $\delta \in (0, 1)$. It is shown by GNQ that Khoudraji’s device provides little asymmetry for Kendall’s $\tau \leq 1/2$, and the maximum asymmetry occurs around $\delta = 1/2$. Our results indicate that under small and moderate $\tau$, the sizes converge to the nominal value as the sample size increases, but with large $\tau$, the sizes are somewhat below the nominal level. All the powers increase as the sample size increases. Compared with Table 3 in GNQ, some of our powers at small sample sizes are not as good as theirs, but our powers with large sample sizes are superior to theirs. This is not surprising, as our test is based on the asymptotic distribution of the test statistics, while GNQ’s test is based on the approximate distribution of bootstrap samples.

We choose five common copulas for the test of radial and joint symmetry, and Table 2 reports the sizes and powers of both tests.

The sizes for both tests are close to 5% for all different $\tau$ even at small sample sizes, and powers exhibit an expected pattern as the sample sizes increase. Since joint symmetry is a more stringent condition than is radial symmetry, it is seen from Table 2 that the powers corresponding to joint symmetry are much higher than those for radial symmetry.

Following Bücher, Dette, and Volgushev (2012), hereinafter BDV12, we use Clayton, Gumbel–Hougaard, two ordinal sum models, $t$-copula, and an asymmetric negative logistic copula model to evaluate the sizes and powers of our associativity test. The details of the two ordinal sum models can be seen in BDV12 (p. 127), and the sizes and powers of our test are listed in Table 3. Overall, the pattern of our results is very similar to the pattern in table 1 in BDV12, even though the sizes for our ordinal sum
models seem more reasonable. In all cases but two, the sizes tend to be smaller than the nominal sizes, which is not detrimental and is similar to the pattern observed in BDV12. Since our test still relies on the asymptotic distribution of the test statistic, our test is more powerful with large samples, although we resort to bootstrapping for the estimation of the asymptotic variance of the test statistic. The test for the other component of Archimedeanity, $\delta_C(u) < u$ versus $\delta_C(u) = u$, completely follows the testing procedure in BDV12. Hence, the Archimedeanity testing results are expected to exhibit a very similar pattern as that in table 1 in BDV12; that is, the sizes are close to the associativity test while the powers are close to 1.00. We omit the Archimedeanity test here as there is nothing new compared to BDV12.

Table 4 summarizes the sizes and powers of the max-stability test that can be compared to Bücher, Dette, and Volgushev (2011). Although the sizes of our test are somewhat off for $n = 200$, they appear to converge to their corresponding nominal levels when $n$ becomes larger. The powers increase dramatically as the sample size increases, particularly if the powers begin with low values with small sample sizes. Again, we observe that the powers of our test with small sample sizes are not impressive compared to table 1 in Bücher, Dette, and Volgushev.

| Table 3. Sizes and powers of the test of associativity in the same setting as Bücher, Dette, and Volgushev (2012). Nominal levels are 5% and 10% |
|---------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                                | $\tau$          | $n = 200$       | $n = 500$       | $n = 1000$      | $n = 200$       | $n = 500$       | $n = 1000$      |
|                                |                 | 5%              | 10%             | 5%              | 10%             | 5%              | 10%             |
| SIZE                           |                 |                 |                 |                 |                 |                 |                 |
| Clayton                        | 1/3             | 0.073           | 0.131           | 0.096           | 0.159           | 0.089           | 0.154           |
| Clayton                        | 2/3             | 0.019           | 0.042           | 0.038           | 0.066           | 0.032           | 0.076           |
| Gumbel                         | 1/3             | 0.056           | 0.104           | 0.093           | 0.170           | 0.101           | 0.181           |
| Gumbel                         | 2/3             | 0.015           | 0.032           | 0.012           | 0.038           | 0.029           | 0.055           |
| Ordinal, $A$                  | 1/3             | 0.012           | 0.023           | 0.022           | 0.034           | 0.027           | 0.059           |
| Ordinal, $A$                  | 2/3             | 0.009           | 0.018           | 0.022           | 0.039           | 0.019           | 0.037           |
| Ordinal, $B$                  | 1/3             | 0.013           | 0.025           | 0.014           | 0.025           | 0.017           | 0.042           |
| Ordinal, $B$                  | 2/3             | 0.013           | 0.023           | 0.011           | 0.029           | 0.016           | 0.033           |
| t(df = 1)                      | 1/3             | 0.570           | 0.701           | 0.976           | 0.988           | 1.000           | 1.000           |
| t(df = 1)                      | 2/3             | 0.166           | 0.260           | 0.658           | 0.773           | 0.986           | 0.994           |
| Aneglog $S/Pi_1$               | 0               | 0.099           | 0.166           | 0.069           | 0.121           | 0.045           | 0.100           |
| Aneglog $S/Pi_1$               | 1/4             | 0.130           | 0.205           | 0.121           | 0.198           | 0.135           | 0.221           |
| Aneglog $S/Pi_1$               | 1/2             | 0.435           | 0.555           | 0.718           | 0.797           | 0.901           | 0.942           |
| Table 4. Sizes and powers of the test of max-stability in the same setting as Bücher, Dette, and Volgushev (2011). Nominal levels are 5% and 10% |
|---------------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
|                                | $\tau$          | $n = 200$       | $n = 500$       | $n = 1000$      | $n = 200$       | $n = 500$       | $n = 1000$      |
|                                |                 | 5%              | 10%             | 5%              | 10%             | 5%              | 10%             |
| S I Z E                        |                 |                 |                 |                 |                 |                 |                 |
| S I Z E                        |                 |                 |                 |                 |                 |                 |                 |
| Gumbel                         | 0.5             | 0.092           | 0.152           | 0.048           | 0.107           | 0.046           | 0.094           |
| Gumbel                         | 0.75            | 0.023           | 0.056           | 0.023           | 0.038           | 0.038           | 0.058           |
| clayton                        | 0.25            | 0.603           | 0.709           | 0.954           | 0.976           | 1.000           | 1.000           |
| clayton                        | 0.5             | 0.947           | 0.969           | 1.000           | 1.000           | 1.000           | 1.000           |
| clayton                        | 0.75            | 0.926           | 0.960           | 1.000           | 1.000           | 1.000           | 1.000           |
| clayton                        | 0.25            | 0.191           | 0.289           | 0.308           | 0.435           | 0.708           | 0.810           |
| clayton                        | 0.5             | 0.260           | 0.391           | 0.625           | 0.743           | 0.975           | 0.989           |
| clayton                        | 0.75            | 0.228           | 0.344           | 0.657           | 0.786           | 0.987           | 0.995           |
| clayton                        | 0.25            | 0.176           | 0.268           | 0.219           | 0.317           | 0.402           | 0.528           |
| clayton                        | 0.5             | 0.165           | 0.243           | 0.303           | 0.435           | 0.685           | 0.795           |
| clayton                        | 0.75            | 0.048           | 0.107           | 0.177           | 0.253           | 0.438           | 0.557           |
The strategy of DATP is particularly important when the data concentrate in a certain area. For example, the data with large $\tau$ usually fall into a narrow band along the diagonal line connecting $(0,0)$ and $(1,1)$ when $d = 2$, and the DATP takes out the testing points that are around the corner of $(0,1)$ and $(1,0)$. Specifically, in our simulations and data analyses, for each grid point, we examine the number of observations falling into a length-0.1 square whose upper-right corner is located at this grid point, and leave out the points that have zero observations. Figure 2 illustrates the DATP in the max-stability test using a random draw of size 200 from a Clayton copula with $\tau = 0.75$. The points and diamonds are the primitive testing points formed by equally spaced grids, while the diamonds are the selected testing points after applying the DATP procedure. With primitive testing points, the powers are 0.807 at level 5% and 0.864 at level 10%, as opposed to the 0.926 and 0.960 powers with the DATP in Table 4 for the Clayton copula and $n = 200$.

5. APPLICATIONS

In this section, we illustrate our methodology on applications to two datasets. In all the analyses, the set $\Lambda$ of testing points, the number of bootstrap samples for the test of associativity, and the choice of values, $r$, for the test of max-stability are the same as in our simulations in Section 4.

5.1 Nutritional Habits Survey Data

This dataset was collected by the U.S. Department of Agriculture in 1985 as part of a survey on nutritional habits of $n = 737$ women with ages ranging from 25 to 50 years. Five variables of daily intakes were measured: calcium (in mg), iron (in mg), protein (in g), vitamin A (in $\mu$g), and vitamin C (in mg). GNQ used a Cramér-von Mises statistic to test for bivariate symmetry of the pairwise copulas. We perform the same test based on our method. The resulting $p$-values are listed in Table 5. The general pattern of $p$-values is similar to the one found by GNQ. However, unlike these authors, our test does not reject a symmetric copula structure at a 5% level for the pairs (calcium, iron), (protein, vitamin A), and (iron, vitamin C), although the $p$-value for the test on (calcium, iron) is really on the boundary. A closer look at the rank plots of those pairs of variables suggests that this conclusion is not unreasonable. Indeed, our conclusion on these three pairs also remains the same when different sets of testing points are used.

5.2 S&P 500 and DAX Return Data

This dataset consist of $n = 396$ observations of two major stock indices during 2009 and 2010: the U.S. American S&P 500 and the German DAX. Following a previous analysis by

<table>
<thead>
<tr>
<th>Table 5. $p$-values of the test of a symmetric copula structure on pairs of variables from the nutritional habits survey data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variable</td>
</tr>
<tr>
<td>Calcium</td>
</tr>
<tr>
<td>Iron</td>
</tr>
<tr>
<td>Protein</td>
</tr>
<tr>
<td>Vitamin A</td>
</tr>
</tbody>
</table>

Figure 2. Illustration of data adaptive testing points (DATP) in the max-stability test. The solid points and diamonds are the primitive testing points, while the diamonds are the testing points after applying the DATP procedure. The online version of this figure is in color.
We have performed simulation experiments to evaluate our test and have concluded that our method is reliable and powerful for assessing common assumptions on the structure of copulas, particularly for moderate sample sizes. We proposed a DATP procedure for the choice of testing points to enhance our method. Both the numerical experiments and data analyses indicated that the testing results are robust to different sets of testing points if chosen appropriately, yet theoretical results for the optimal number and placements of the testing points are still worth further investigation. Although there is a possibility that certain characteristics of the copula process may hold only at a few points, and this could lead to power loss if those certain characteristics are in the null space and those few points form our set of testing points, this will unlikely occur if the set of points is chosen to be representative of the whole copula process. We have not detected such a problem in our simulation studies. Since our test is developed based on the asymptotic distribution of empirical copula processes, the small sample properties of our test may not be optimal compared to the bootstrap-based testing methods. The demand for sample size in our testing approach varies depending on the particular copula structure to be examined. The test of equality between two copulas of Rémillard and Scaillet (2009) for paired samples could also be used to test certain copula structures.

We have also tried our test on the Marshall–Olkin family, which is not differentiable at all \((u_1, u_2) \in [0, 1]^2\), and we found that the procedure gave satisfactory testing results based on unreported simulation experiments. This suggests that our method may only require the differentiability of \(C(u_1, u_2)\) in a neighborhood of \((u_1, u_2) \in A\) but not in the entire unit square \([0, 1]^2\).

We have illustrated our testing approach on two different datasets, and we expect that our methodology will find widespread use in applications. Some computer code of the tests can be obtained upon request from the first author.

**APPENDIX**

**Proof of Theorem 3.1.** Let \(d = 2\) and let \((u, v, w) \in [0, 1]^3\). We study the properties of the estimators for \(C[C(u, v), w]\) or \(C[u, C(v, w)]\) that appear in the associativity property of Archimedean copulas. Without loss of generality, we only consider the form \(C[C(u, v), w]\) below for convenience, but the derivation can be easily adapted to \(C[u, C(v, w)]\) and the asymptotic results remain the same. Let \(\hat{\alpha}(u, v, w) = C(u, v, w)\) and \(\alpha(u, v, w) = \{\hat{D}_n[\hat{D}_n(u, v), w]\} \setminus \{\hat{D}_n[\hat{D}_n(u, v), w]\}\). Theorem 2.2 in BDV12 immediately yields

\[
\sqrt{n}(\hat{\alpha}(u, v, w) - \alpha(u, v, w)) \xrightarrow{d} U^C[C(u, v), w] - \frac{\partial C(u, v)}{\partial u} U^C[C(u, v), 1]
\]

\[
- \frac{\partial C(u, v)}{\partial v} \bigg|_{u=C(u,v), v=1} U^C(u, 1) + \frac{\partial C(u, v)}{\partial u} \bigg|_{u=C(u,v), v=1} U^C(u, v) - \frac{\partial C(u, v)}{\partial u} U^C(u, 1)
\]

\[
- \frac{\partial C(u, v)}{\partial v} U^C(u, 1),
\]

as \(n \to \infty\). Let \(H_0 = [C(u_1, v_1), w_1], \ldots, C(u_n, v_n)]^T\) and \(\tilde{H}_n = \{\hat{D}_n[\hat{D}_n(u_1, v_1), w_1], \ldots, \hat{D}_n[\hat{D}_n(u_n, v_n), w_n]\}^T\). Then we have

\[
\sqrt{n}(\tilde{H}_n - H_0) \xrightarrow{d} N(0, \Psi),
\]

where \(\Psi\) is a covariance matrix. We have illustrated our testing approach on two different datasets, and we expect that our methodology will find widespread use in applications. Some computer code of the tests can be obtained upon request from the first author.
for a covariance matrix $\Psi$, as $n \to \infty$. Now let $\phi(u, v, w) = \widehat{C}_n(C_a(u, v, w))$. Following the same arguments as in the proof of theorem 2.2 in BDV12, we obtain
\[
\sqrt{n} \left[ \phi(u, v, w) - \phi_0(u, v, w) \right] \xrightarrow{d} U^C[C(u, v), w] + \frac{dC(u, v)}{u} \bigg|_{u=C(u, v), v} U^C(u, v).
\]

Let $G_0 = \{G_1, \ldots, G_\ell\}$ and $\widetilde{G}_n = \{C_1(C_a(u_1, v_1), w_1), \ldots, C_\ell(C_a(u_\ell, v_\ell), w_\ell)\}^T$. Then we have
\[
\sqrt{n} \left[ \widetilde{G}_n - G_0 \right] \xrightarrow{d} N(0, \Phi),
\]
for a covariance matrix $\Phi$, as $n \to \infty$. The form of $\Psi$ and $\Phi$ can be easily derived from $\text{cov}(U^C(u), U^C(u)) = C(u \wedge u) - C(u)C(u)$. Since we suggest to employ the bootstrap method to estimate those covariance matrices due to their intricate form and we have successfully used such method to estimate $\Psi$, we omit the lengthy expression of $\Psi$ and $\Phi$ here. 

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