Nonparametric Identification of Copula Structures

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We propose a unified framework for testing a variety of assumptions commonly made about the structure of copulas, including symmetry, radial symmetry, joint symmetry, associativity and Archimedeanity, and max-stability. Our test is nonparametric and based on the asymptotic distribution of the empirical copula process. We perform simulation experiments to evaluate our test and conclude that our method is reliable and powerful for assessing common assumptions on the structure of copulas, particularly when the sample size is moderately large. We illustrate our testing approach on two datasets.

KEY WORDS: Archimedeanity; Associativity; Asymptotic normality; Max-stability; Symmetry; Test.

1. INTRODUCTION

Modeling multivariate data with copulas has been used in many applications in recent years, especially in actuarial science, finance, hydrology, and survival analysis; see Genest and Favre (2007) for a review and references therein. In particular, the use of copulas in financial areas such as asset pricing and credit risk management has soared; see, for example, the books by Cherubini, Luciano, and Vecchiato (2004) and by McNeil, Frey, and Embrechts (2005). The apparent success of copulas is due to their ability to model the dependence structure of a random vector separately from its marginal behavior. Indeed, because of the representation theorem provided by Sklar (1959), every multivariate cumulative distribution function, F, of a continuous random vector, $\mathbf{X} = (X_1, \dots, X_d)^{\mathrm{T}}$ on $\mathbb{R}^d$, can be written as

$$\begin{aligned} F(x_1, \dots, x_d) &= \Pr(X_1 \le x_1, \dots, X_d \le x_d) \\ &= C\{F_1(x_1), \dots, F_d(x_d)\}, \end{aligned} \tag{1}$$

where $F_j(x_j) = \Pr(X_j \le x_j)$, $x_j \in \mathbb{R}$, are the continuous marginal distributions, $j = 1, \dots, d$, and $C : [0, 1]^d \to [0, 1]$ is the unique copula. The latter contains all the information about the dependence structure of the random vector, $\mathbf{X}$, and can be obtained from

$$\begin{aligned} C(u_1, \dots, u_d) &= \Pr(U_1 \le u_1, \dots, U_d \le u_d) \\ &= F\left\{F_1^{-1}(u_1), \dots, F_d^{-1}(u_d)\right\}, \end{aligned} \tag{2}$$

where $F_j^{-1}(u_j) = \inf\{x | F_j(x) \ge u_j\}, u_j \in [0, 1]$, is the quantile function of F_j for $j = 1, \dots, d$, and $\mathbf{U} = (U_1, \dots, U_d)^{\mathrm{T}}$ with $U_j = F_j(X_j)$, $j = 1, \dots, d$. From (2), the copula C is itself a cumulative distribution function, of $\mathbf{U}$, on the d-dimensional unit hypercube with uniform marginal distributions.

From a practical point of view, the representation in Equation (1) allows the dependence structure to be modeled given the marginal distributions by choosing one of the many parametric copula models available in the literature; see, for example, the books by Joe (1997), Nelsen (2006), and Panjer (2006) for the most common parametric families of copulas. Mikosch (2006) then asked the following natural question: How does one choose a copula? In those days, copula models were often chosen by subject-matter specialists for mathematical convenience rather than usefulness for the data at hand. The main purpose of this article is to change this procedure by proposing a new and nonparametric method that allows the data to choose the copula model.

One model selection approach that has recently received sustained attention is to consider goodness-of-fit tests for copulas. The procedure consists of testing that the unknown copula, C, belongs to a parametric family, $\mathcal{C}_0 = \{C_\theta | \theta \in \Theta \subset \mathbb{R}^p, p \ge 1\}$, of copulas, that is, $H_0 : C \in \mathcal{C}_0$ versus $H_1 : C \notin \mathcal{C}_0$. Tests aimed at discriminating between H_0 and H_1 have been proposed; see the reviews by Genest, Rémillard, and Beaudoin (2009), Berg (2009), and references therein. These methods consider the univariate marginal distributions as infinite-dimensional nuisance parameters, and typically in that context, the observations are replaced by the maximally invariant statistics, that is, the ranks. Other attempts for choosing parametric families of copulas can be found in the literature. For example, Craiu and Craiu (2008) investigated choices based on various distances between a nonparametric density estimate and the density implied by parametric copula models fitted by maximum likelihood. Michiels and De Schepper (2008) introduced test space models for copulas, that is, comparable families of copulas with respect to their dependence range.

Our approach is different from the previous ones because we aim at testing the *structure* of copulas. Our test can be viewed as complementary to the above goodness-of-fit methods because it will suggest sensible classes, $\mathcal{C}_0$, of parametric families of copulas. Very recently, attempts to specify copula structures emerged in several articles. Jaworski (2010) proposed a simple test for the associativity structure based on the asymptotic distribution of the pointwise copula estimator. The disadvantage of this test is that it only tests the associativity at a particular point rather than for the whole copula process; see the discussion in Bücher, Dette, and Volgushev (2012). Bücher, Dette, and Volgushev (2012) derived Cramér-von Mises and Kolmogorov–Smirnov type test statistics for assessing the characteristics of associativity, based on which they further formed a test statistic for Archimedeanity. Bücher, Dette, and Volgushev (2011) developed a test for extreme-value dependence through

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the unique property of the minimum weighted L^2-distance of extreme-value copulas. A test for bivariate symmetry of copulas based on Cramér-von Mises and Kolmogorov–Smirnov functionals of the rank-based empirical copula process was introduced by Genest, Nešlehová, and Quessy (2012), but it does not provide more specific guidance for model choice (see also Kojadinovic and Yan 2012). Other specific tests of structures have been reviewed by Genest and Nešlehová (2012, 2013; see also references therein). In contrast, we propose a unified framework for testing a variety of assumptions commonly made for the structure of copulas, including symmetry, radial symmetry, joint symmetry, associativity and Archimedeanity, and max-stability. Our test is nonparametric and based on the asymptotic distribution of the empirical copula process.

The article is organized as follows. In Section 2, we describe various structures of copulas that are of interest to applications, such as symmetries, Archimedeanity, max-stability, and others. In particular, we provide a novel schematic representation of those structures and interrelations for various popular copula models in Figure 1. In Section 3, we introduce our nonparametric testing procedures based on the asymptotic distribution of the empirical copula. We report the results of Monte Carlo simulations in Section 4 where we investigate the empirical sizes and powers of our tests, as well as the choice of testing points. In Section 5, we present illustrations of our methodology on two applications. The article ends with a discussion in Section 6.

2. COPULA STRUCTURES

2.1 Copulas With Symmetry Structures

For the sake of simplicity, we follow Nelsen (1993) and focus on the case $d = 2$, although the results can be extended to $d > 2$. There are various concepts of bivariate symmetry for copulas. The simplest notion is to say that a copula, C, is *symmetric* if

$$C(u_1, u_2) - C(u_2, u_1) = 0, \tag{3}$$

for all $(u_1, u_2) \in [0, 1]^2$. Many copulas satisfy this structure of symmetry. Moreover, two random variables are exchangeable if and only if their marginal distributions are the same and their copula is symmetric. Another concept of symmetry is to say that a copula, C, is *radially symmetric* if

$$C(u_1, u_2) - C(1 - u_1, 1 - u_2) + 1 - u_1 - u_2 = 0, \tag{4}$$

for all $(u_1, u_2) \in [0, 1]^2$. There exist copulas that are symmetric but not radially symmetric and, conversely, copulas that are radially symmetric but not symmetric. In particular, exchangeability does not imply radial symmetry and vice versa. Finally, we say that a copula, C, is *jointly symmetric* if

$$C(u_1, u_2) + C(u_1, 1 - u_2) - u_1 = 0$$

and

$$C(u_1, u_2) + C(1 - u_1, u_2) - u_2 = 0, \tag{5}$$

for all $(u_1, u_2) \in [0, 1]^2$. Jointly symmetric copulas are necessarily radially symmetric, but they could be symmetric or not. Figure 1 depicts these notions of symmetry structures and their interrelations. More discussions on bivariate symmetry concepts can be found in Nelsen (1993).

2.2 Copulas With Archimedean Structures

Archimedean copulas are an important class of copulas. Their popularity arises from their mathematical convenience and attractive properties. Specifically, a copula is *Archimedean* if it takes the form

$$C_\phi(u_1, \ldots, u_d) = \phi^{[-1]}\{\phi(u_1) + \cdots + \phi(u_d)\}, \tag{6}$$

for all $(u_1, \ldots, u_d) \in [0, 1]^d$ and for some convex, continuous and strictly decreasing function $\phi : [0, 1] \to [0, \infty]$ such that $\phi(1) = 0$. The function, ϕ, is called the generator of the copula, for which the (pseudo) inverse generator, $\phi^{[-1]}(t)$, must be a completely monotonic function. This general class encompasses many well-known copulas; see for instance Genest and MacKay (1986a,b), Joe (1997), and Nelsen (2006).

Archimedean copulas are symmetric and *associative*, that is, for $d = 2$, $C(u_1, u_2) = C(u_2, u_1)$ and $C\{C(u_1, u_2), u_3\} = C\{u_1, C(u_2, u_3)\}$ for all $(u_1, u_2, u_3) \in [0, 1]^3$. Indeed, an Archimedean copula is characterized by an associative copula with $\delta_C(u) < u$ for all $u \in (0, 1)$, where $\delta_C(u) = C(u, u)$ is the diagonal section of a copula and $\delta_C(u) = \phi^{[-1]}\{2\phi(u)\}$ for an Archimedean copula, C, with generator ϕ, see Ling (1965).

2.3 Copulas With Max-Stable Structures

A copula, C, is said to be *max-stable*, or equivalently to be an *extreme-value copula*, if

$$C(u_1, \ldots, u_d) - C^r\left(u_1^{1/r}, \ldots, u_d^{1/r}\right) = 0, \tag{7}$$

for all $(u_1, \ldots, u_d) \in [0, 1]^d$ and for any $r > 0$. Such copulas appear naturally as the copula $C(u_1, \ldots, u_d) = \lim_{n\to\infty} \tilde{C}^n (u_1^{1/n}, \ldots, u_d^{1/n})$ associated with componentwise maxima of n independent and identically distributed random vectors with copula $\tilde{C}$.

A related structure is that a copula, C, is *homogeneous* of degree $k \in \mathbb{R}$, that is,

$$C(\lambda u_1, \ldots, \lambda u_d) - \lambda^k C(u_1, \ldots, u_d) = 0,$$

for all $(u_1, \ldots, u_d) \in [0, 1]^d$ and for any $0 \le \lambda \le 1$. An interesting result is that if C is a homogeneous copula of degree k, then necessarily $1 \le k \le d$ and C has to belong to the Cuadras–Augé family with parameter $\theta = d - k$, which are symmetric and max-stable. The proof of this result for $d = 2$ can be found in Nelsen (2006, p. 102) and its extension to $d > 2$ is straightforward.

2.4 Relations Between Copula Structures

Copulas satisfy a version of the Fréchet–Hoeffding bounds inequality. Specifically, when $d = 2$, for every copula C and every $(u_1, u_2) \in [0, 1]^2$:

$$W(u_1, u_2) \le C(u_1, u_2) \le M(u_1, u_2),$$

where $W(u_1, u_2) = \max(u_1 + u_2 - 1, 0)$ and $M(u_1, u_2) = \min(u_1, u_2)$ are themselves copulas, and both are symmetric and radially symmetric. In addition, W is Archimedean and M is max-stable. Although these two copulas are too simplistic for real applications, they play a major theoretical role as bounds. Another simple yet important copula is the product copula $\Pi(u_1, u_2) = u_1 u_2$ that corresponds to the independence case. It satisfies all aforementioned notions of symmetries and is both Archimedean and max-stable.

The elliptical copulas (Fang, Fang, and Kotz 2002) are those associated with an elliptically contoured distribution, F, in (2) (Fang, Kotz, and Ng 1990). They are symmetric and radially

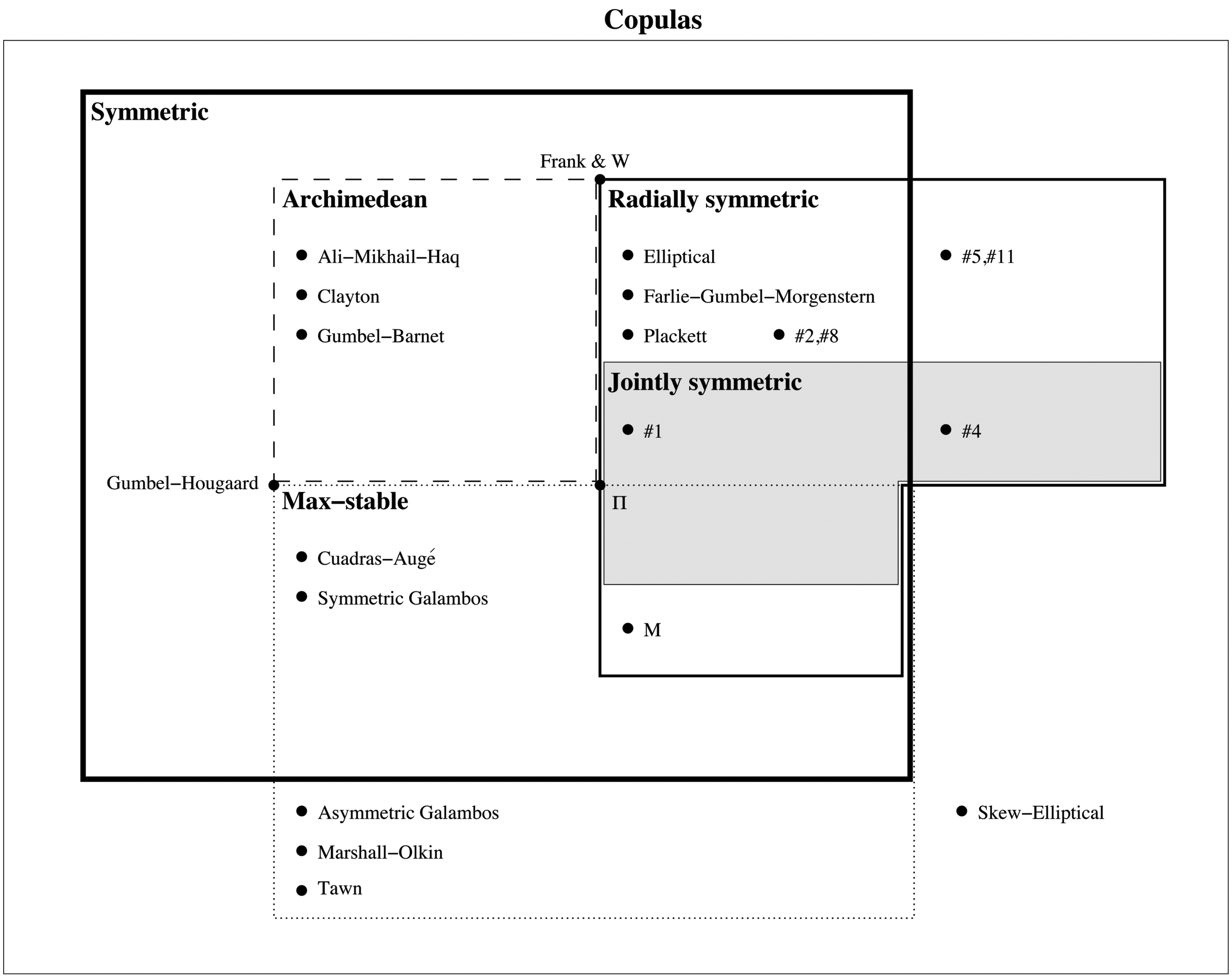

Figure 1. Schematic representation of various copula structures and their interrelations for $d = 2$: symmetric, radially symmetric, jointly symmetric, Archimedean, and max-stable. The names and reference numbers of some of the most popular copulas (as described in Section 2.4) are included according to their structures. The online version of this figure is in color.

symmetric. Two important members of this family are the Gaussian copula and the Student's t copula (Demarta and McNeil 2005). Skew-elliptical copulas are those associated with skew-elliptical distributions, see the book edited by Genton (2004) for an overview. As their name suggests, they are asymmetric copulas. In particular, skew-t copulas have been studied by Kollo and Pettere (2010). Other asymmetric copulas have been constructed by Liebscher (2008).

The Farlie–Gumbel–Morgenstern copula (Farlie 1960; Gumbel 1958; Morgenstern 1956) and the Plackett copula (Plackett 1965) are both symmetric and radially symmetric. Nelsen (1993) studied examples of copulas with different symmetry structures. In his figure 1, copulas #2 and #8 are both symmetric and radially symmetric, and copula #1 is in addition also jointly symmetric. Copulas #5 and #11 are only radially symmetric, whereas copula #4 is both radially and jointly symmetric. These copulas are presented in our Figure 1 for $d = 2$.

Some of the most important members of the family of Archimedean copulas are the Clayton copula (Clayton 1978), the Frank copula (Frank 1979), the Gumbel–Hougaard copula (Gumbel 1960), the Ali-Mikhail-Haq copula (Ali, Mikhail, and Haq 1978), and the Gumbel–Barnet copula (Gumbel 1960; Barnett 1980). They are all symmetric by the nature of the Archimedean form. The Frank copula is also radially symmetric. The Gumbel–Hougaard copula satisfies the max-stable condition as well. In fact, it is the only copula that is both Archimedean and max-stable (Nelsen 2006, p. 143).

Several copulas are symmetric and max-stable, including the Cuadras–Augé family (Cuadras and Augé 1981) and the symmetric Galambos copula (Galambos 1975), although there is also an asymmetric version. Two copulas that are asymmetric and max-stable are the Marshall–Olkin family of copulas (Marshall and Olkin 1967), also known as the generalized Cuadras–Augé family, and the Tawn copula (Tawn 1988) of which the symmetric Gumbel–Hougaard copula is a special case.

3. TESTING PROCEDURES

3.1 Asymptotics

Let $\mathbf{u} = (u_1, \ldots, u_d)^{\mathrm{T}}$ and let Λ be a set of user-chosen points in the unit hypercube $[0, 1]^d$ of cardinality c. Typically, Λ is a regular discretization of the interior of $[0, 1]^d$, and the appropriate choice of c will be investigated in Section 4.2. Define $\mathbf{G} = \{C(\mathbf{u}) : \mathbf{u} \in \Lambda\}$ to be the length, c, column vector of the copula, C, evaluated at the points in Λ.

Given the marginals, $\mathbf{U}$, we can choose a moment estimator for $C(\mathbf{u})$ such that

$$\widehat{C}_n(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{d} I(U_{ki} \le u_k),$$

where $I(\cdot)$ denotes the indicator function. Let $\widehat{\mathbf{G}}_n = \{\widehat{C}_n(\mathbf{u}) : \mathbf{u} \in \Lambda\}$. It has been shown (e.g., Tsukahara 2005; Rémillard and Scaillet 2009; Genest and Segers 2010) that

$$\sqrt{n}\{\widehat{C}_n(\mathbf{u}) - C(\mathbf{u})\} \xrightarrow{d} U^C(\mathbf{u}),$$

as $n \to \infty$, where $U^C(\mathbf{u})$ is a d-dimensional pinned C-Brownian sheet; that is, it is a centered Gaussian random field with $\mathrm{cov}\{U^C(\mathbf{u}_i), U^C(\mathbf{u}_j)\} = C(\mathbf{u}_i \wedge \mathbf{u}_j) - C(\mathbf{u}_i)C(\mathbf{u}_j)$, where $\mathbf{u}_i \wedge \mathbf{u}_j$ denotes the componentwise minimum. Then, we have

$$\sqrt{n}(\widehat{\mathbf{G}}_n - \mathbf{G}) \xrightarrow{d} N_c(\mathbf{0}, \boldsymbol{\Sigma}),$$

as $n \to \infty$, where

$$\boldsymbol{\Sigma}_{ij} = \mathrm{cov}\{U^C(\mathbf{u}_i), U^C(\mathbf{u}_j)\} = C(\mathbf{u}_i \wedge \mathbf{u}_j) - C(\mathbf{u}_i)C(\mathbf{u}_j). \quad (8)$$

Given a known copula, we can directly calculate $\boldsymbol{\Sigma}_{ij}$. If the form of the copula is unknown, we choose to use the plug-in estimator by replacing $C(\mathbf{u})$ with its moment estimator $\widehat{C}_n(\mathbf{u})$.

It is often the case that the marginals, $\mathbf{U}$, are unknown and we only observe the data points $\mathbf{X} = (X_1, \ldots, X_d)^{\mathrm{T}}$. For each $\mathbf{X}_i = (X_{1i}, \ldots, X_{di})^{\mathrm{T}}$, $i = 1, \ldots, n$, we can estimate its corresponding $\mathbf{U}_i = (U_{1i}, \ldots, U_{di})^{\mathrm{T}}$ by the moment estimators $\widehat{\mathbf{U}}_i = (\widehat{U}_{1i}, \ldots, \widehat{U}_{di})^{\mathrm{T}}$, where $\widehat{U}_{ki} = \frac{1}{n}\sum_{j=1}^{n} I(X_{kj} \le X_{ki})$ for $k = 1, \ldots, d$. Then, we can estimate $C(\mathbf{u})$ again by the moment estimator but based on $\widehat{\mathbf{U}}_i$, $i = 1, \ldots, n$. Let

$$\widehat{D}_n(\mathbf{u}) = \frac{1}{n}\sum_{i=1}^{n}\prod_{k=1}^{d} I(\widehat{U}_{ki} \le u_k).$$

It has been shown (Tsukahara 2005; Rémillard and Scaillet 2009; Genest and Segers 2010) that when $C(\mathbf{u})$ is differentiable with continuous partial derivatives $\partial C(\mathbf{u})/\partial u_k$ for $k = 1, \ldots, d$,

$$\sqrt{n}\{\widehat{D}_n(\mathbf{u}) - C(\mathbf{u})\} \xrightarrow{d} U^C(\mathbf{u}) - \sum_{k=1}^{d}\frac{\partial C(\mathbf{u})}{\partial u_k} U^C(\mathbf{1}, u_k, \mathbf{1}),$$

as $n \to \infty$, where $\mathbf{1}$ is a vector of ones with the proper length depending on the current context. Define $\widehat{\mathbf{H}}_n = \{\widehat{D}_n(\mathbf{u}) : \mathbf{u} \in \Lambda\}$ and $\mathbf{H} = \mathbf{G}$. Then, it can be derived that

$$\sqrt{n}(\widehat{\mathbf{H}}_n - \mathbf{H}) \xrightarrow{d} N_c(\mathbf{0}, \boldsymbol{\Pi}),$$

as $n \to \infty$, where

$$\begin{aligned}\boldsymbol{\Pi}_{ij} &= C(\mathbf{u}_i \wedge \mathbf{u}_j) - C(\mathbf{u}_i)C(\mathbf{u}_j) \\ &\quad - \sum_{k=1}^{d}\frac{\partial C(\mathbf{u}_j)}{\partial u_{kj}}[C\{\mathbf{u}_i \wedge (\mathbf{1}, u_{kj}, \mathbf{1})\} - u_{kj}C(\mathbf{u}_i)] \\ &\quad - \sum_{k=1}^{d}\frac{\partial C(\mathbf{u}_i)}{\partial u_{ki}}[C\{\mathbf{u}_j \wedge (\mathbf{1}, u_{ki}, \mathbf{1})\} - u_{ki}C(\mathbf{u}_j)] \\ &\quad + \sum_{m=1}^{d}\sum_{n=1}^{d}\frac{\partial C(\mathbf{u}_i)}{\partial u_{mi}}\frac{\partial C(\mathbf{u}_j)}{\partial u_{nj}}[C\{(\mathbf{1}, u_{mi}, \mathbf{1}) \wedge (\mathbf{1}, u_{nj}, \mathbf{1})\} \\ &\quad - u_{mi}u_{nj}]. \end{aligned} \quad (9)$$

We again choose the plug-in method for estimating $\boldsymbol{\Pi}$. The unknown $C(\mathbf{u})$ can be estimated by $\widehat{D}_n(\mathbf{u})$, and we estimate the partial derivatives following the technique in Rémillard and Scaillet (2009). Specifically, let $\mathbf{e}_k$ be a length d vector of zeroes but with the kth element replaced by 1, $\partial C(\mathbf{u})/\partial u_k$ can be consistently estimated by $\{\widehat{D}_n(\mathbf{u} + h_n\mathbf{e}_k) - \widehat{D}_n(\mathbf{u} - h_n\mathbf{e}_k)\}/(2h_n)$. We choose $h_n = n^{-1/4}$ following Bücher, Dette, and Volgushev (2012). Genest and Segers (2010) had shown that $\boldsymbol{\Pi} \le \boldsymbol{\Sigma}$ componentwise under some moderate conditions of positive dependence.

To assess the Archimedean property for $d = 2$, we have to estimate $C\{C(u_1, u_2), u_3\}$ and $C\{u_1, C(u_2, u_3)\}$ by $\widehat{C}_n\{\widehat{C}_n(u_1, u_2), u_3\}$ and $\widehat{C}_n\{u_1, \widehat{C}_n(u_2, u_3)\}$ or by $\widehat{D}_n\{\widehat{D}_n(u_1, u_2), u_3\}$ and $\widehat{D}_n\{u_1, \widehat{D}_n(u_2, u_3)\}$, respectively. Thus, we also investigate the asymptotics of those estimators. We denote $\widehat{\mathbb{G}}_n = \{\widehat{C}_n\{\widehat{C}_n(u_1, u_2), u_3\} : (u_1, u_2, u_3)^{\mathrm{T}} \in \Lambda\}$ for known marginals, and $\widehat{\mathbb{H}}_n = \{\widehat{D}_n\{\widehat{D}_n(u_1, u_2), u_3\} : (u_1, u_2, u_3)^{\mathrm{T}} \in \Lambda\}$ for unknown marginals. Let $\mathbb{G}_0 = \{C\{C(u_1, u_2), u_3\} : (u_1, u_2, u_3)^{\mathrm{T}} \in \Lambda\}$ and $\mathbb{H}_0 = \mathbb{G}_0$. The elements in $\widehat{\mathbb{G}}_n$ can also be in the form of $\widehat{C}_n\{u_1, \widehat{C}_n(u_2, u_3)\}$, and in $\widehat{\mathbb{H}}_n$, it can be $\widehat{D}_n\{u_1, \widehat{D}_n(u_2, u_3)\}$. Correspondingly, the elements in $\mathbb{G}_0$ and $\mathbb{H}_0$ will have $C\{u_1, C(u_2, u_3)\}$. We prove the following result for $d = 2$ in the Appendix.

Theorem 3.1. If $\partial C(u_1, u_2)/\partial u_1$ and $\partial C(u_1, u_2)/\partial u_2$ exist and are continuous at all $(u_1, u_2) \in [0, 1]^2$, then for any Λ that is a set of $(u, v, w) \in [0, 1]^3$, we have

$$\sqrt{n}(\widehat{\mathbb{G}}_n - \mathbb{G}_0) \xrightarrow{d} N_c(\mathbf{0}, \boldsymbol{\Phi}),$$

and

$$\sqrt{n}(\widehat{\mathbb{H}}_n - \mathbb{H}_0) \xrightarrow{d} N_c(\mathbf{0}, \boldsymbol{\Psi}),$$

as $n \to \infty$, for covariance matrices $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$. The dimension c of the multinormal distribution is determined by the cardinality of Λ consisting of triplets.

Under unknown margins, Genest and Segers (2009) showed the consistency and asymptotic normality of rank-based estimators of the Pickands dependence function, which can be expressed as appropriate functionals of the empirical copula. Segers (2012) further demonstrated the weak convergence of the empirical copula process without requiring the existence or continuity of the partial derivatives on certain boundaries. The result in Theorem 3.1 is closely related to Genest and Segers (2009) and Segers (2012), but here we consider the asymptotics of empirical copulas at again empirical copulas rather than at constant probabilities. These results are also related to Bücher, Dette, and Volgushev (2012) who considered a particular empirical copula process based on the characteristics of associativity. In Theorem 3.1, the explicit forms of the two matrices, $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$, are intricate as discussed in the Appendix. In practice, we therefore estimate them via bootstrap techniques. We can use a standard bootstrap procedure to estimate $\boldsymbol{\Phi}$, but when marginals are unknown, we resort to the multiplier bootstrap provided in Bücher, Dette, and Volgushev (2012) to estimate $\boldsymbol{\Psi}$.

3.2 Test Statistics

We now describe the statistics to test the various copula structures described in Section 2. For simplicity of exposition, we describe the bivariate setting of $d = 2$. The symmetry structure (3) and the max-stability structure (7) of copulas can be rewritten in vector form:

$$(1, -1)\begin{pmatrix} C(u_1, u_2) \\ C(u_2, u_1) \end{pmatrix} = 0$$

and

$$(1, -1)\begin{pmatrix} C(u_1, u_2) \\ C^r\left(u_1^{1/r}, u_2^{1/r}\right) \end{pmatrix} = (1, -1)\,\mathbf{f}\begin{pmatrix} C(u_1, u_2) \\ C\left(u_1^{1/r}, u_2^{1/r}\right) \end{pmatrix} = 0,$$

for a function, $\mathbf{f}$. The radial and joint symmetry structures (4) and (5), respectively, only require obvious modifications of the above representation. For example, radial symmetry uses

$$(1, -1)\begin{pmatrix} C(u_1, u_2) + 1 - u_1 - u_2 \\ C(1 - u_1, 1 - u_2) \end{pmatrix} = 0.$$

Since the equality has to hold for any $(u_1, u_2) \in [0, 1]^2$ if the copula possesses the corresponding particular property, we extend the equalities to $\mathbf{Af}(\mathbf{G}) = \mathbf{0}$ for a contrast matrix $\mathbf{A}$. Indeed, many properties can be written into such forms. For those properties, we can perform the hypothesis test by the following test statistic

$$\mathrm{TS1} = n\{\mathbf{Af}(\widehat{\mathbf{G}}_n)\}^{\mathrm{T}}(\mathbf{AB}^{\mathrm{T}}\boldsymbol{\Sigma}\mathbf{BA}^{\mathrm{T}})^{-1}\{\mathbf{Af}(\widehat{\mathbf{G}}_n)\},$$

if the marginals, $\mathbf{U}$, are known. The contrast matrix $\mathbf{A}$ has the row rank q and the function $\mathbf{f} = (f_1, \ldots, f_s)^{\mathrm{T}}$. The matrix $\mathbf{B}$ is defined as $\mathbf{B}_{ij} = \partial f_j / \partial \mathbf{G}_i$, $i = 1, \ldots, c$, $j = 1, \ldots, s$. When the marginals, $\mathbf{U}$, need to be estimated from $\mathbf{X}$, we use the test statistic

$$\mathrm{TS2} = n\{\mathbf{Af}(\widehat{\mathbf{H}}_n)\}^{\mathrm{T}}(\mathbf{AB}^{\mathrm{T}}\boldsymbol{\Pi}\mathbf{BA}^{\mathrm{T}})^{-1}\{\mathbf{Af}(\widehat{\mathbf{H}}_n)\}.$$

Due to the asymptotic normality of $\widehat{\mathbf{G}}_n$ and $\widehat{\mathbf{H}}_n$, $\mathrm{TS1} \xrightarrow{d} \chi_q^2$ and $\mathrm{TS2} \xrightarrow{d} \chi_q^2$ asymptotically. The choice of $\mathbf{A}$ and $\mathbf{f}$ depends on the individual tests, and both $\boldsymbol{\Sigma}$ and $\boldsymbol{\Pi}$ are estimated by the plug-in method.

If the hypothesis of symmetry structure is rejected, then the copula cannot be Archimedean. If it is not rejected, then the identification of the Archimedean copula structure (6) requires two tests as mentioned in Section 2.2: (i) Associativity and (ii) $\delta_C(u) < u$.

Associativity requires $C\{C(u_1, u_2), u_3\} = C\{u_1, C(u_2, u_3)\}$ for all $(u_1, u_2, u_3) \in [0, 1]^3$, which can be rewritten as

$$(1, -1)\begin{pmatrix} C\{C(u_1, u_2), u_3\} \\ C\{u_1, C(u_2, u_3)\} \end{pmatrix} = 0.$$

Then,

$$\mathrm{TS3} = n(\mathbf{A}\widehat{\mathbb{G}}_n)^{\mathrm{T}}(\mathbf{A}\boldsymbol{\Phi}\mathbf{A}^{\mathrm{T}})^{-1}(\mathbf{A}\widehat{\mathbb{G}}_n)$$

can be used for testing the associativity when the marginals are known. For unknown marginals, we have the following test:

$$\mathrm{TS4} = n(\mathbf{A}\widehat{\mathbb{H}}_n)^{\mathrm{T}}(\mathbf{A}\boldsymbol{\Psi}\mathbf{A}^{\mathrm{T}})^{-1}(\mathbf{A}\widehat{\mathbb{H}}_n).$$

The asymptotic covariance matrices, $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$, are defined in the Appendix. However, for practical purposes, in the associativity test, we estimate $\boldsymbol{\Phi}$ and $\boldsymbol{\Psi}$ using bootstrap resampling techniques due to their intricate form as discussed in the previous section.

Bücher, Dette, and Volgushev (2012) proposed a function, $A_n(\widehat{D}_n) = \max\{\eta(1-\eta) : \widehat{D}_n(\eta, \eta) = \eta\}$ for $0 < \eta < 1$, to examine $\delta_C(u) < u$ versus $\delta_C(u) = u$. Under mild conditions, $A_n(\widehat{D}_n) = o_p(n^{-\alpha})$ for any $\alpha < 1/2$ if $\delta_C(u) < u$ and $A_n(\widehat{D}_n) \geq \eta(1-\eta) + o_p(1)$ if there exists an η such that $C(\eta, \eta) = \eta$. They integrated $A_n(\widehat{D}_n)$ with their test for associativity to form the test statistic for assessing Archimedeanity. We can naturally borrow the same idea to combine $A_n(\widehat{D}_n)$ with our test for associativity into the following test statistic for Archimedeanity assessment:

$$\mathrm{TS5} = \mathrm{TS4} + k_n \psi\{A_n(\widehat{D}_n)\},$$

where $k_n \sim n^{\alpha}$, $\alpha \in (0, 1/2)$ is a user-chosen constant and $\psi(\cdot)$ is an increasing function with $\psi(0) = 0$.

The matrix $\mathbf{A}$ in all test statistics needs to be of full row rank to avoid the singularity problem. This is equivalent to requiring that all the contrasts formed by $\mathbf{A}$ be linearly independent. Particular attention is necessary for both the radial symmetry and joint symmetry tests because their corresponding contrasts formed over a set of testing points can possibly overlap. For example, the following contrasts for testing joint symmetry are linearly dependent contrasts because if any three of them hold, then the remaining one holds too:

$$\begin{aligned} &C(u_1, u_2) + C(u_1, 1 - u_2) - u_1 = 0; \\ &C(u_1, 1 - u_2) + C(1 - u_1, 1 - u_2) - 1 + u_2 = 0; \\ &C(u_1, u_2) + C(1 - u_1, u_2) - u_2 = 0; \\ &C(1 - u_1, u_2) + C(1 - u_1, 1 - u_2) - 1 + u_1 = 0. \end{aligned}$$

There are virtually only three free contrasts, and thus the matrix $\mathbf{A}$ needs to filter out one of them to maintain full row rank.

4. MONTE CARLO SIMULATIONS

We conduct simulations to evaluate our testing procedures for various structures of copulas and meanwhile compare our method to several other tests that have been developed for assessing particular structures. Since the marginals are rarely known, we conduct all the simulations with unknown marginals. Specifically, we examine the tests for symmetry, radial symmetry, joint symmetry, associativity, and max-stability. All the simulations have 1000 replicates, and the nominal level of all tests is 5% unless otherwise specified. We choose $r = 2$ in testing the max-stable structure, and we choose 500 bootstrap samples in the associativity test. We compare our symmetry test to Genest, Nešlehová, and Quessy (2012), the associativity test to Bücher, Dette, and Volgushev (2012), and the max-stability test to Bücher, Dette, and Volgushev (2011). In these comparisons, we follow their choice of copula models and simulation settings.

4.1 Testing Results

Table 1 reports the sizes and powers of the test of symmetry. Following Genest, Nešlehová, and Quessy (2012), hereinafter GNQ, three copula models are made asymmetric by Khoudraji's device (Khoudraji 1995). Specifically, an asymmetric version of a copula, $C(u_1, u_2)$, is defined at all $(u_1, u_2) \in [0, 1]^2$ by

$$K_\delta(u_1, u_2) = u_1^{\delta} C\left(u_1^{1-\delta}, u_2\right),$$

Table 1. Sizes and powers of the test of symmetry in the same setting as Genest, Nešlehová, and Quessy (2012)

			Clayton n			Gaussian n			Gumbel n		
	δ	τ	250	500	1000	250	500	1000	250	500	1000
SIZE	0	1/4	0.154	0.098	0.078	0.164	0.096	0.063	0.154	0.112	0.075
		1/2	0.076	0.057	0.037	0.069	0.047	0.039	0.068	0.053	0.056
		3/4	0.013	0.003	0.008	0.009	0.005	0.001	0.002	0.004	0.005
	0	0*	0.160	0.104	0.075	—	—	—	—	—	—
POWER	1/4	0.5	0.251	0.299	0.607	0.174	0.222	0.349	0.200	0.262	0.477
		0.7	0.813	0.990	1.000	0.495	0.865	1.000	0.610	0.929	0.999
		0.9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	1/2	0.5	0.344	0.381	0.616	0.323	0.434	0.739	0.460	0.689	0.958
		0.7	0.835	0.997	1.000	0.892	0.997	1.000	0.971	1.000	1.000
		0.9	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	3/4	0.5	0.235	0.189	0.230	0.322	0.378	0.634	0.474	0.670	0.940
		0.7	0.423	0.572	0.882	0.726	0.945	0.999	0.839	0.997	1.000
		0.9	0.938	0.999	1.000	0.995	1.000	1.000	0.990	1.000	1.000

*The sizes for the independent copula Π ($\delta = \tau = 0$).

for $\delta \in (0, 1)$. It is shown by GNQ that Khoudraji's device provides little asymmetry for Kendall's $\tau \leq 1/2$, and the maximum asymmetry occurs around $\delta = 1/2$. Our results indicate that under small and moderate τ, the sizes converge to the nominal value as the sample size increases, but with large τ, the sizes are somewhat below the nominal level. All the powers increase as the sample size increases. Compared with Table 3 in GNQ, some of our powers at small sample sizes are not as good as their powers, but our powers with large sample sizes are superior to theirs. This is not surprising, as our test is based on the asymptotic distribution of the test statistics, while GNQ's test is based on the approximate distribution of bootstrap samples.

We choose five common copulas for the test of radial and joint symmetry, and Table 2 reports the sizes and powers of both tests. The sizes for both tests are close to 5% for all different τ even at small sample sizes, and powers exhibit an expected pattern as the sample sizes increase. Since joint symmetry is a more stringent condition than is radial symmetry, it is seen from Table 2 that the powers corresponding to joint symmetry are much higher than those for radial symmetry.

Following Bücher, Dette, and Volgushev (2012), hereinafter BDV12, we use Clayton, Gumbel–Hougaard, two ordinal sum models, t-copula, and an asymmetric negative logistic copula model to evaluate the sizes and powers of our associativity test. The details of the two ordinal sum models can be seen in BDV12 (p. 127), and the sizes and powers of our test are listed in Table 3. Overall, the pattern of our results is very similar to the pattern in table 1 in BDV12, even though the sizes for our ordinal sum

Table 2. Sizes and powers of the test of radial and joint symmetry

			Radial				Joint		
	τ		$n = 250$	$n = 500$	$n = 1000$		$n = 250$	$n = 500$	$n = 1000$
Π	0		0.057	0.067	0.058	SIZE	0.077	0.083	0.063
Frank	1/4	SIZE	0.066	0.072	0.065	POWER	0.995	1.000	1.000
	1/2		0.080	0.073	0.056		1.000	1.000	1.000
	3/4		0.027	0.038	0.049		1.000	1.000	1.000
Gaussian	1/4		0.088	0.064	0.050		0.989	1.000	1.000
	1/2		0.091	0.058	0.056		1.000	1.000	1.000
	3/4		0.026	0.045	0.031		1.000	1.000	1.000
Clayton	1/4	POWER	0.372	0.624	0.930		0.993	1.000	1.000
	1/2		0.801	0.983	1.000		1.000	1.000	1.000
	3/4		0.861	0.999	1.000		1.000	1.000	1.000
Gumbel	1/4		0.118	0.164	0.307		0.882	0.996	1.000
	1/2		0.216	0.358	0.681		1.000	1.000	1.000
	3/4		0.178	0.378	0.754		1.000	1.000	1.000

Table 3. Sizes and powers of the test of associativity in the same setting as Bücher, Dette, and Volgushev (2012). Nominal levels are 5% and 10%

			$n = 200$		$n = 500$		$n = 1000$	
		τ	5%	10%	5%	10%	5%	10%
SIZE	Clayton	1/3	0.073	0.131	0.096	0.159	0.089	0.154
	Clayton	2/3	0.019	0.042	0.038	0.066	0.032	0.076
	Gumbel	1/3	0.056	0.104	0.093	0.170	0.101	0.181
	Gumbel	2/3	0.015	0.032	0.012	0.038	0.029	0.055
	Ordinal_A	1/3	0.012	0.023	0.022	0.034	0.027	0.059
	Ordinal_A	2/3	0.009	0.018	0.022	0.039	0.019	0.037
	Ordinal_B	1/3	0.013	0.025	0.014	0.025	0.017	0.042
	Ordinal_B	2/3	0.013	0.023	0.011	0.029	0.016	0.033
POWER	$t(\text{df} = 1)$	1/3	0.570	0.701	0.976	0.988	1.000	1.000
	$t(\text{df} = 1)$	2/3	0.166	0.260	0.658	0.773	0.986	0.994
	Aneglog	1/4	0.130	0.205	0.121	0.198	0.135	0.221
	Aneglog	1/2	0.435	0.555	0.718	0.797	0.901	0.942

models seem more reasonable. In all cases but two, the sizes tend to be smaller than the nominal sizes, which is not detrimental and is similar to the pattern observed in BDV12. Since our test still relies on the asymptotic distribution of the test statistic, our test is more powerful with large samples, although we resort to bootstrapping for the estimation of the asymptotic variance of the test statistic. The test for the other component of Archimedeanity, $\delta_C(u) < u$ versus $\delta_C(u) = u$, completely follows the testing procedure in BDV12. Hence, the Archimedeanity testing results are expected to exhibit a very similar pattern as that in table 1 in BDV12; that is, the sizes are close to the associativity test while the powers are close to 1.000. We omit the Archimedeanity test here as there is nothing new compared to BDV12.

Table 4 summarizes the sizes and powers of the max-stability test that can be compared to Bücher, Dette, and Volgushev (2011). Although the sizes of our test are somewhat off for $n = 200$, they appear to converge to their corresponding nominal levels when n becomes larger. The powers increase dramatically as the sample size increases, particularly if the powers begin with low values with small sample sizes. Again, we observe that the powers of our test with small sample sizes are not impressive compared to table 1 in Bücher, Dette, and Volgushev

Table 4. Sizes and powers of the test of max-stability in the same setting as Bücher, Dette, and Volgushev (2011). Nominal levels are 5% and 10%

			$n = 200$		$n = 500$		$n = 1000$	
		τ	5%	10%	5%	10%	5%	10%
SIZE	Π	0	0.099	0.166	0.069	0.121	0.045	0.100
	Gumbel	0.25	0.109	0.172	0.067	0.124	0.053	0.093
	Gumbel	0.5	0.092	0.152	0.048	0.107	0.046	0.094
	Gumbel	0.75	0.023	0.056	0.023	0.038	0.038	0.058
POWER	Clayton	0.25	0.603	0.709	0.954	0.976	1.000	1.000
	Clayton	0.5	0.947	0.969	1.000	1.000	1.000	1.000
	Clayton	0.75	0.926	0.960	1.000	1.000	1.000	1.000
	Frank	0.25	0.191	0.289	0.308	0.435	0.708	0.810
	Frank	0.5	0.260	0.391	0.625	0.743	0.975	0.989
	Frank	0.75	0.228	0.344	0.657	0.786	0.987	0.995
	Gaussian	0.25	0.176	0.268	0.219	0.317	0.402	0.528
	Gaussian	0.5	0.165	0.243	0.303	0.435	0.685	0.795
	Gaussian	0.75	0.048	0.107	0.177	0.253	0.438	0.557

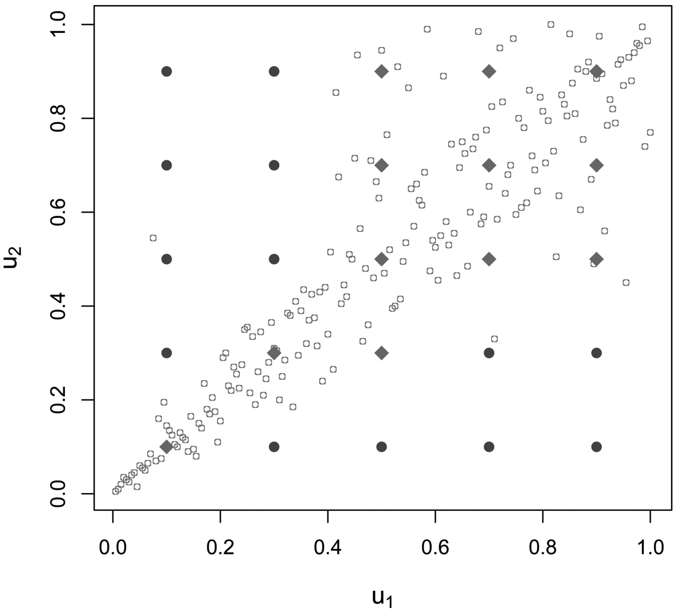

Figure 2. Illustration of data adaptive testing points (DATP) in the max-stability test. The solid points and diamonds are the primitive testing points, while the diamonds are the testing points after applying the DATP procedure. The online version of this figure is in color.

(2011), but they become satisfactory with large samples due to the asymptotic characteristics of the test statistics.

4.2 The Size of Λ

The choice of testing points in the set Λ can introduce uncertainty to the testing results. It is often the case that the characteristics of a copula hold at any point in $[0, 1]^d$. In principle, the test should therefore maintain the power even if it is performed at only one point. However, from a practical point of view, it would be inefficient to perform the test only at a small number of points due to the randomness of real observations. In general, a larger number of testing points lead to greater power, but excessively many testing points can destroy the size of the test. The deterioration of sizes associated with a large number of testing points is likely because more testing points introduce stronger correlations between the empirical copulas at those points, and the estimation of all such correlations is difficult given a sample of moderate size. In sum, there is a trade-off between the size and the power when we choose the set Λ of testing points. In our experience, we find that approximately 10–30 contrasts work well if the testing points represent the coverage of data well.

We choose a primitive design of testing points in Λ as the grid expanded by an equally spaced sequence in [0, 1], for the purpose of making testing points cover the whole region. Then, we propose a data adaptive testing points (DATP) procedure to enhance our testing method, in particular for data with large Kendall's τ. The idea of DATP is that we choose only the testing points that anchor in a region of data abundance, while we abandon the ones in areas of data scarcity as those points typically provide no additional information but simply add noise to the test statistics.

The strategy of DATP is particularly important when the data concentrate in a certain area. For example, the data with large τ usually fall into a narrow band along the diagonal line connecting (0,0) and (1,1) when $d = 2$, and the DATP takes out the testing points that are around the corner of (0,1) and (1,0). Specifically, in our simulations and data analyses, for each grid point, we examine the number of observations falling into a length-0.1 square whose upper-right corner is located at this grid point, and leave out the points that have zero observations. Figure 2 illustrates the DATP in the max-stability test using a random draw of size 200 from a Clayton copula with $\tau = 0.75$. The points and diamonds are the primitive testing points formed by equally spaced grids, while the diamonds are the selected testing points after applying the DATP procedure. With primitive testing points, the powers are 0.807 at level 5% and 0.864 at level 10%, as opposed to the 0.926 and 0.960 powers with the DATP in Table 4 for the Clayton copula and $n = 200$.

5. APPLICATIONS

In this section, we illustrate our methodology on applications to two datasets. In all the analyses, the set Λ of testing points, the number of bootstrap samples for the test of associativity, and the choice of values, r, for the test of max-stability are the same as in our simulations in Section 4.

5.1 Nutritional Habits Survey Data

This dataset was collected by the U.S. Department of Agriculture in 1985 as part of a survey on nutritional habits of $n = 737$ women with ages ranging from 25 to 50 years. Five variables of daily intakes were measured: calcium (in mg), iron (in mg), protein (in g), vitamin A (in μg), and vitamin C (in mg). GNQ used a Cramér-von Mises statistic to test for bivariate symmetry of the pairwise copulas. We perform the same test based on our method. The resulting p-values are listed in Table 5. The general pattern of p-values is similar to the one found by GNQ. However, unlike these authors, our test does not reject a symmetric copula structure at a 5% level for the pairs (calcium, iron), (protein, vitamin A), and (iron, vitamin C), although the p-value for the test on (calcium, iron) is really on the boundary. A closer look at the rank plots of those pairs of variables suggests that this conclusion is not unreasonable. Indeed, our conclusion on these three pairs also remains the same when different sets of testing points are used.

5.2 S&P 500 and DAX Return Data

This dataset consist of $n = 396$ observations of two major stock indices during 2009 and 2010: the U.S. American S&P 500 and the German DAX. Following a previous analysis by

Table 5. p-values of the test of a symmetric copula structure on pairs of variables from the nutritional habits survey data

Variable	Calcium	Iron	Protein	Vitamin A	Vitamin C
Calcium		0.061	0.013	0.002	0.591
Iron			0.766	0.025	0.086
Protein				0.254	0.123
Vitamin A					0.813

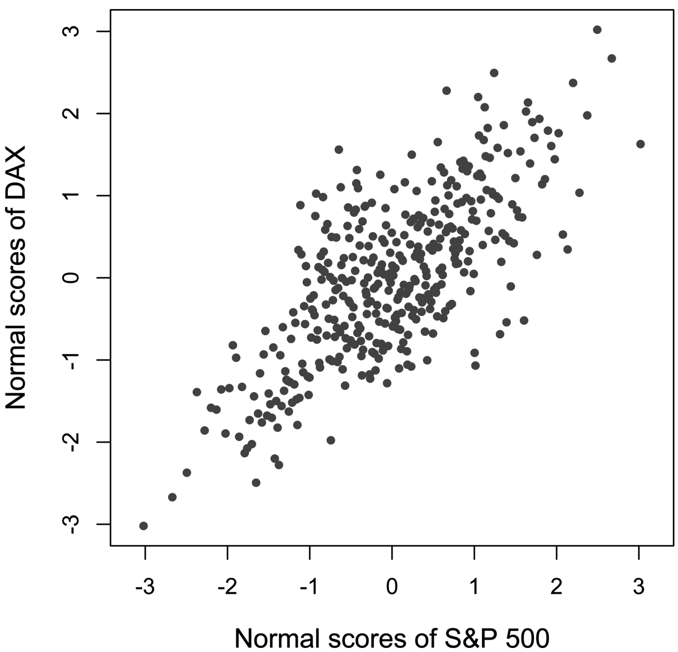

Figure 3. Scatterplot of normal scores for S&P 500 and DAX returns. The online version of this figure is in color.

Brechmann and Schepsmeier (2013), each time series is filtered using an ARMA(1,1)-GARCH(1,1) model with Student's t innovations and standardized residuals transformed nonparametrically to copula data using the corresponding empirical distribution functions. The goal is to identify the underlying copula structure to propose a suitable parametric copula model for the dependence of those two indices.

The scatterplot of normal scores of the S&P 500 and the DAX is shown in Figure 3. The p-values of our tests of symmetry, radial symmetry, joint symmetry, max-stability, and associativity are 0.172, 0.082, 0.000, 0.009, and 0.113, respectively. The results suggest that, at the 5% level, symmetry, radial symmetry, and associativity of the copula structures cannot be rejected. The Archimedeanity test based on BDV12 also fails to reject this property even at the 10% level. The only copula that has those three structures is the Frank copula, see Figure 1. This data example illustrates well the guidance provided by our method for selecting a parametric copula model. For instance, the Student's t copula, one of the common elliptical copulas, is symmetric and radially symmetric, but not Archimedean, and thus is screened out.

6. DISCUSSION

In this article, we have proposed a unified framework for testing a variety of assumptions commonly made for the structure of copulas, including symmetry, radial symmetry, joint symmetry, associativity and Archimedeanity, and max-stability. Our test is nonparametric and based on the asymptotic distribution of the empirical copula process. Other structures could be tested as well, as long as they can be formulated by $\mathbf{Af}(\mathbf{G}) = \mathbf{0}$ as in Section 3.2. For example, it is known that $\delta_C(u) = \delta_M(u)$ for all $u \in [0, 1]$ implies that $C = M$ and $C(u, 1 - u) = 0$ for all $u \in [0, 1]$ also implies $C = W$. Obviously, these conditions can be written into linear contrasts and thus tested by our method.

We have performed simulation experiments to evaluate our test and have concluded that our method is reliable and powerful for assessing common assumptions on the structure of copulas, particularly for moderate sample sizes. We proposed a DATP procedure for the choice of testing points to enhance our method. Both the numerical experiments and data analyses indicated that the testing results are robust to different sets of testing points if chosen appropriately, yet theoretical results for the optimal number and placements of the testing points are still worth further investigation. Although there is a possibility that certain characteristics of the copula process may hold only at a few points, and this could lead to power loss if those certain characteristics are in the null space and those few points form our set of testing points, this will unlikely occur if the set of points is chosen to be representative of the whole copula process. We have not detected such a problem in our simulation studies. Since our test is developed based on the asymptotic distribution of empirical copula processes, the small sample properties of our test may not be optimal compared to the bootstrap-based testing methods. The demand for sample size in our testing approach varies depending on the particular copula structure to be examined. The test of equality between two copulas of Rémillard and Scaillet (2009) for paired samples could also be used to test certain copula structures.

We have also tried our test on the Marshall–Olkin family, which is not differentiable at all $(u_1, u_2) \in [0, 1]^2$, and we found that the procedure gave satisfactory testing results based on unreported simulation experiments. This suggests that our method may only require the differentiability of $C(u_1, u_2)$ in a neighborhood of $(u_1, u_2) \in \Lambda$ but not in the entire unit square $[0, 1]^2$.

We have illustrated our testing approach on two different datasets, and we expect that our methodology will find widespread use in applications. Some computer code of the tests can be obtained upon request from the first author.

APPENDIX

Proof of Theorem 3.1. Let $d = 2$ and let $(u, v, w) \in [0, 1]^3$. We study the properties of the estimators for $C\{C(u, v), w\}$ or $C\{u, C(v, w)\}$ that appear in the associativity property of Archimedean copulas. Without loss of generality, we only consider the form $C\{C(u, v), w\}$ below for convenience, but the derivation can be easily adapted to $C\{u, C(v, w)\}$ and the asymptotic results remain the same. Let $\alpha_0(u, v, w) = C\{C(u, v), w\}$ and $\alpha_n(u, v, w) = \{\widehat{D}_n\{\widehat{D}_n(u, v), w\}$. Theorem 2.2 in BDV12 immediately yields

$$
\begin{aligned}
&\sqrt{n}\{\alpha_n(u, v, w) - \alpha_0(u, v, w)\}\\
&\xrightarrow{d} U^C\{C(u, v), w\} - \frac{\partial C(u, v)}{\partial u}\Big|_{u=C(u,v)} U^C\{C(u, v), 1\}\\
&\quad - \frac{\partial C(u, v)}{\partial v}\Big|_{v=w} U^C(1, w)\\
&\quad + \frac{\partial C(u, v)}{\partial u}\Big|_{u=C(u,v)} \left\{ U^C(u, v) - \frac{\partial C(u, v)}{\partial u} U^C(u, 1)\right.\\
&\quad \left. - \frac{\partial C(u, v)}{\partial v} U^C(1, v) \right\},
\end{aligned}
$$

as $n \to \infty$. Let $\mathbb{H}_0 = \{C\{C(u_1, v_1), w_1\}, \ldots, C\{C(u_c, v_c), w_c\}\}^{\mathrm{T}}$ and $\widehat{\mathbb{H}}_n = \{\widehat{D}_n\{\widehat{D}_n(u_1, v_1), w_1\}, \ldots, \widehat{D}_n\{\widehat{D}_n(u_c, v_c), w_c\}\}^{\mathrm{T}}$. Then we have

$$\sqrt{n}(\widehat{\mathbb{H}}_n - \mathbb{H}_n) \xrightarrow{d} N_c(\mathbf{0}, \boldsymbol{\Psi}),$$

for a covariance matrix $\boldsymbol{\Psi}$, as $n \to \infty$. Now let $\alpha_n'(u, v, w) = \widehat{C}_n\{\widehat{C}_n(u, v), w\}$. Following the same arguments as in the proof of theorem 2.2 in BDV12, we obtain

$$\sqrt{n}\{\alpha_n'(u, v, w) - \alpha_0(u, v, w)\} \xrightarrow{d} U^C\{C(u, v), w\} + \frac{\partial C(u, v)}{\partial u}|_{u=C(u,v)} U^C(u, v).$$

Let $\mathbb{G}_0 = \mathbb{H}_0$ and $\widehat{\mathbb{G}}_n = \{\widehat{C}_n\{\widehat{C}_n(u_1, v_1), w_1\}, \ldots, \widehat{C}_n\{\widehat{C}_n(u_c, v_c), w_c\}\}^{\mathrm{T}}$. Then we have

$$\sqrt{n}(\widehat{\mathbb{G}}_n - \mathbb{G}_0) \xrightarrow{d} N_c(\mathbf{0}, \boldsymbol{\Phi}),$$

for a covariance matrix $\boldsymbol{\Phi}$, as $n \to \infty$. The form of $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ can be easily derived from $\mathrm{cov}\{U^C(\mathbf{u}_i), U^C(\mathbf{u}_j)\} = C(\mathbf{u}_i \wedge \mathbf{u}_j) - C(\mathbf{u}_i)C(\mathbf{u}_j)$. Since we suggest to employ the bootstrap method to estimate those covariance matrices due to their intricate form and we have successfully used such method to estimate $\boldsymbol{\Psi}$, we omit the lengthy expression of $\boldsymbol{\Psi}$ and $\boldsymbol{\Phi}$ here. $\square$

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REFERENCES

Ali, M., Mikhail, N., and Haq, S. (1978), "A Class of Bivariate Distributions Including the Bivariate Logistics," *Journal of Multivariate Analysis*, 8, 405–412. [668]

Barnett, V. (1980), "Some Bivariate Uniform Distributions," *Communications in Statistics—Theory and Methods*, 9, 453–461. [668]

Berg, D. (2009), "Copula Goodness-Of-Fit Testing: An Overview and Power Comparison," *The European Journal of Finance*, 15, 675–701. [666]

Brechmann, E. C., and Schepsmeier, U. (2013), "Modeling Dependence With C- and D-Vine Copulas: The R Package CDVine," *Journal of Statistical Software*, 52, 1–27. [674]

Bücher, A., Dette, H., and Volgushev, S. (2011), "New Estimators of the Pickands Dependence Function and a Test for Extreme-Value Dependence," *The Annals of Statistics*, 39, 1963–2006. [666,670,672]

——— (2012), "A Test for Archimedeanity in Bivariate Copula Models," *Journal of Multivariate Analysis*, 110, 121–132. [666,669,670,671]

Cherubini, U., Luciano, E., and Vecchiato, W. (2004), *Copula Methods in Finance*, New York: Wiley. [666]

Clayton, D. (1978), "A Model for Association in Bivariate Life Tables and Its Application in Epidemiological Studies of Familial Tendency in Chronic Disease Incidence," *Biometrika*, 65, 141–151. [668]

Craiu, M., and Craiu, R. V. (2008), "Choice of Parametric Families of Copulas," *Advances and Applications in Statistics*, 10, 25–40. [666]

Cuadras, C. M., and Augé, J. (1981), "A Continuous General Multivariate Distribution and its Properties," *Communications in Statistics—Theory and Methods*, 10, 339–353. [668]

Demarta, S., and McNeil, A. (2005), "The t Copula and Related Copulas," *International Statistical Review*, 73, 111–129. [668]

Fang, H. B., Fang, K. T., and Kotz, S. (2002), "The Meta-Elliptical Distributions With Given Marginals," *Journal of Multivariate Analysis*, 82, 1–16. [667]

Fang, K.-T., Kotz, S., and Ng, K.-W. (1990), *Symmetric Multivariate and Related Distributions*, Monographs on Statistics and Applied Probability 36, London: Chapman and Hall. [667]

Farlie, D. J. G. (1960), "The Performance of Some Correlation Coefficients for a General Bivariate Distribution," *Biometrika*, 47, 307–323. [668]

Frank, M. J. (1979), "On the Simultaneous Associativity of $f(x)$ and $x + y - f(x, y)$," *Aequationes Mathematicae*, 19, 194–226. [668]

Galambos, J. (1975), "Order Statistics of Samples From Multivariate Distributions," *Journal of the American Statistical Association*, 70, 674–680. [668]

Genest, C., and Favre, A.-C. (2007), "Everything You Always Wanted to Know About Copula Modeling but Were Afraid to Ask," *Journal of Hydrologic Engineering*, 12, 347–368. [666]

Genest, C., and MacKay, J. (1986a), "The Joy of Copulas: Bivariate Distributions With Uniform Marginals," *The American Statistician*, 40, 280–283. [667]

Genest, C., and MacKay, R. (1986b), "Copules Archimédiennes et Familles de Lois Bidimensionelles Dont les Marges Sont Données," *The Canadian Journal of Statistics*, 14, 145–159. [667]

Genest, C., Nešlehová, J., and Quessy, J.-F. (2012), "Tests of Symmetry for Bivariate Copulas," *Annals of the Institute of Statistical Mathematics*, 64, 811–834. [667,671]

Genest, C., and Nešlehová, J. (2012), "Copula Modeling for Extremes," *Encyclopedia of Environmetrics, Second Edition*, eds. A. H. El-Shaarawi and W. W. Piegorsch, Chichester, UK: Wiley, pp. 530–541. [667]

——— (2013). "Assessing and Modeling Asymmetry in Bivariate Continuous Data," In *Copulae in Mathematical and Quantitative Finance, Proceedings of the Workshop Held in Cracow*, 10–11 July 2012, eds. P. Jaworski, F. Durante, W. K. Härdle, Berlin: Springer, pp. 91–114. [667]

Genest, C., Rémillard, B., and Beaudoin, D. (2009), "Goodness-Of-Fit Tests for Copulas: A Review and a Power Study," *Insurance: Mathematics and Economics*, 44, 199–213. [666]

Genest, C., and Segers, J. (2009), "Rank-Based Inference for Bivariate Extreme-Value Copulas," *The Annals of Statistics*, 37, 2990–3022. [669]

——— (2010), "On the Covariance of the Asymptotic Empirical Copula Process," *Journal of Multivariate Analysis*, 101, 1837–1845. [669]

Genton, M. G. (2004), *Skew-Elliptical Distributions and Their Applications: A Journey Beyond Normality*, Edited Volume, Boca Raton: Chapman & Hall/CRC. [668]

Gumbel, E. J. (1958), *Statistics of Extremes*, New York: Columbia University Press. [668]

——— (1960), "Distributions des Valeurs Extrêmes en Plusieurs Dimensions," *Publications de l'Institut de Statistique de l'Université de Paris*, 9, 171–173. [668]

Jaworski, P. (2010), "Testing Archimedeanity," in *Combining Soft Computing & Statistics Methods*, eds. C. Borgelt, G. González-Rodriguez, W. Trutschnig, M. Lubiano, M. Gil, P. Grzegorzewski, O. Hryniewicz, AISC 77, Berlin: Springer-Verlag, pp. 353–360. [666]

Joe, H. (1997), *Multivariate Models and Dependence Concepts*, London: Chapman & Hall. [666,667]

Khoudraji, A. (1995), "Contributions à l'étude des copules et à la modélisation de valeurs extrêmes bivariées," Ph.D. thesis, Université Laval, Québec, Canada. [670]

Kojadinovic, I., and Yan, J. (2012), "A Nonparametric Test of Exchangeability for Extreme-value and Left-tail Decreasing Bivariate Copulas," *Scandinavian Journal of Statistics*, 39, 480–496. [667]

Kollo, T., and Pettere, G. (2010), "Parameter Estimation and Application of the Multivariate Skew t-Copula," in *Copula Theory and Its Applications*, eds. P. Jaworski, F. Durante, W. Haerdle, and T. Rychlik, Lecture Notes in Statistics, Heidelberg: Springer, pp. 198, 289–298. [668]

Liebscher, E. (2008), "Construction of Asymmetric Multivariate Copulas," *Journal of Multivariate Analysis*, 99, 2234–2250. Erratum: *Journal of Multivariate Analysis*, 102, 869–870. [668]

Ling, C.-H. (1965), "Representation of Associative Functions," *Publicationes Mathematicae Debrecen*, 12, 189–212. [667]

Marshall, A., and Olkin, I. (1967), "A Multivariate Exponential Distribution," *Journal of the American Statistical Association*, 62, 30–44. [668]

McNeil, A. J., Frey, R., and Embrechts, P. (2005), *Quantitative Risk Management: Concepts, Techniques, and Tools*, Princeton: Princeton University Press. [666]

Michiels, F., and De Schepper, A. (2008), "Copula Test Space Model—How to Avoid the Wrong Copula Choice," *Kybernetika*, 44, 864–878. [666]

Mikosch, T. (2006), "Copulas: Tales and Facts" (with discussion and rejoinder), *Extremes*, 9, 3–62. [666]

Morgenstern, D. (1956), "Einfache Beispiele Zweidimensionaler Verteilungen," *Mitteilingsblatt für Mathematische Statistik*, 8, 234–235. [668]

Nelsen, R. B. (1993), "Some Concepts of Bivariate Symmetry," *Journal of Nonparametric Statistics*, 3, 95–101. [667,668]

——— (2006), *An Introduction to Copulas* (2nd ed.), New York: Springer. [666,667,668]

Panjer, H. H. (2006), *Operational Risks: Modeling Analytics*, Hoboken, NJ: Wiley. [666]

Plackett, R. L. (1965), "A Class of Bivariate Distributions," *Journal of the American Statistical Association*, 60, 516–522. [668]

Rémillard, B., and Scaillet, O. (2009), "Testing for Equality Between Two Copulas," *Journal of Multivariate Analysis*, 100, 377–386. [669]

Segers, J. (2012), "Asymptotics of Empirical Copula Processes Under Nonrestrictive Smoothness Assumptions," *Bernoulli*, 18, 764–782. [669]

Sklar, A. (1959), "Fonctions de Répartition à n Dimensions et leurs Marges," *Publications of the Institute of Statistics of the University of Paris*, 8, 229–231. [666]

Tawn, J. (1988), "Bivariate Extreme Value Theory: Models and Estimation," *Biometrika*, 75, 397–415. [668]

Tsukahara, H. (2005), "Semiparametric Estimation in Copula Models," *Canadian Journal of Statistics*, 33, 357–375. [669]